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ZETA INTEGRALS OF GSp(4) VIA BESSEL MODELS

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Abstract

Using the Piatetski-Shapiro theory of zeta integrals via Bessel models in [9], we explicitly calculate L-factors of irreducible admissible representations of $\mathrm{GSp}(4, F)$, where F is a non-archimedean local field of characteristic zero.

Contents

Chapter 1

Introduction

An irreducible, admissible representation of an algebraic reductive group over a local field is called *generic* if it has a Whittaker model. Whittaker models are one of the main tools to define local and global L-functions and ε-factors of representations. The theory was developed by Jacquet and Langlands for $\mathrm{GL}(2)$ following ideas of Tate's thesis for $\mathrm{GL}(1)$. The general case of $\mathrm{GL}(n)$ was developed in a series of works by Jacquet, Piatetski-Shapiro and Shalika. It is well-known that any infinite dimensional irreducible, admissible representation of $\mathrm{GL}(2)$ is generic.

Let F be a non-archimedean local field of characteristic zero. In [14], Takloo-Bighash computed L-functions for all generic representations of the group $\mathrm{GSp}(4, F)$. It is similar to the theory of $\mathrm{GL}(n)$ in that the approach is based on the existence of Whittaker models and zeta integrals. The method was first introduced by Novodvorsky in the Corvallis conference [8]. However, it turns out that there are many irreducible, admissible representations of $\mathrm{GSp}(4, F)$ which are not generic.

In the 1970's, Novodvorsky and Piatetski-Shapiro introduced the concept of

Bessel models. In contrast to Whittaker models, every irreducible, admissible, infinite-dimensional representation of $\mathrm{GSp}(4,F)$ admits a Bessel model of some kind; see Theorem 6.1.4 of [11]. Piatetski-Shapiro in [9] defined a new type of zeta integral with respect to Bessel models which led to a parallel method to the $\mathrm{GL}(2)$ case of defining local factors. However, some of the results of [9] were only sketched, and not many factors were calculated explicitly.

Danisman calculated many Piatetski-Shapiro L-factors explicitly in the case of non-split Bessel models. In [5], representations were treated whose Jacquet module with respect to the Siegel parabolic has at most length 2. In [7], this was extended to length at most 3. Non-generic supercuspidals were the topic of [6].

In this work we revisit both Piatetski-Shapiro's original theory and Danisman's explicit calculations. We generalize the theory of [9] in that we do not restrict ourselves to unitary representations. We also fill in some of the missing proofs of [9], for example in the argument that generic representations do not admit "exceptional poles".

Generalizing Danisman's approach, we give a unified treatment of the asymptotics of Bessel functions in the non-split case which works for all representations. The key here is to consider a new type of finite-dimensional module $V_{N,T,\Lambda}$ associated to an irreducible, admissible representation (π, V) of $\mathrm{GSp}(4,F)$. These *Jacquet-Waldspurger modules* control the asymptotics of Bessel functions. Table 3.3 contains the semisimplifications of all Jacquet-Waldspurger modules, and Table 4.1 contains their precise algebraic structure as $F^\times$-modules. A key lemma in the non-split case is due to Danisman; see Proposition 4.1.6.

Regarding the split Bessel models, our first step is to determine the algebraic structure of the Jacquet-Waldspurger modules. In contrast to the non-split case, the Waldspurger functor, in general, is not exact in the split case. We introduce the category $\mathcal{C}$, see (3.2.32), which is a subcategory of smooth representations of $\mathrm{GL}(2,F)$. The category $\mathcal{C}$ is the key tool to calculate the Jacquet-Waldspurger modules of generic representations. We also make use of our calculations of Waldspurger modules of reducible principal series and Lemma 3.2.4 to obtain the Jacquet-Waldspurger modules of non-generic representaions. To determine the asymptotic behavior of Bessel functions, we separate our method into two different cases which are non-generic and generic representations. More specifically, we consider the dimension of twisted Jacquet module $V_{N,\psi}$, that is, $\dim V_{N,\psi}$ is either 0,1, 2, or $+\infty$. If $\dim V_{N,\psi} = 0$, Bessel models do not exist. We are in the non-generic case if $\dim V_{N,\psi} \in \{1,2\}$. Otherwise, (π, V) is generic. In the non-generic case, we prove the similar result to Proposition 4.1.6 of the non-split case which characterizes the connection between the Jacquet Waldspurger modules and the asymptotic behavior of Bessel functions. On the other hand, the generic representations require our deep understanding of the algebraic structure of the HN'-module $V_{N_0,T,\Lambda}$. More explicitly, $V_{N_0,T,\Lambda}$ can be written as a sum of an irreducible HN'-module V_0 and $\oplus[\nu^{3/2}\chi_i](n_i)$ of the Jacquet Waldspurger module. It turns out that $V_0 + [\nu^{3/2}\chi_i](n_i)$ is not a direct sum if and only if $L(s,\pi)^{n_i}$ is a factor of $L^{\mathrm{PS}}_{\mathrm{reg}}(s,\pi,\mu)$. However, our method in the split case needs the restriction to the condition of the character $\Lambda = (\Lambda_1, \Lambda_2)$ in such a way that Λ_1 is not special as described in Table 4.3.

Once the asymptotics are known, it is easy to calculate the *regular part* $L^{\mathrm{PS}}_{\mathrm{reg}}(s,\pi,\mu)$ of the Piatetski-Shapiro L-factor; see Table 4.5 and 4.6. Our results show that in all generic cases $L^{\mathrm{PS}}_{\mathrm{reg}}(s,\pi,\mu)$ coincides with the usual spin Euler factor defined via the local Langlands correspondence, but for non-generic representations these factors generally disagree. The results of Table 4.5 also imply that $L^{\mathrm{PS}}_{\mathrm{reg}}(s, \pi, \mu)$ is independent of the choice of Bessel model in the non-split case.

Chapter 2

Bessel models

2.1 Notations

Let F be a non-archimedean local field of characteristic zero. Let $\mathfrak{o}$ be its ring of integers, $\mathfrak{p}$ the maximal ideal of $\mathfrak{o}$, and ϖ a generator of $\mathfrak{p}$. Let q be the cardinality of $\mathfrak{o}/\mathfrak{p}$. We fix a non-trivial character ψ of F. Let v be the normalized valuation on F, and let ν or $|\cdot|$ be the normalized absolute value on F. Hence $\nu(x) = q^{-v(x)}$ for $x \in F^\times$.

Let $\mathrm{GSp}(4, F) := \{g \in \mathrm{GL}(4, F) : {}^t gJg = \lambda J, \text{ for some } \lambda = \lambda(g) \in F^\times\}$ be defined with respect to the symplectic form

$$J = \begin{bmatrix} & 1_2 \\ -1_2 & \end{bmatrix}. \tag{2.1.1}$$

Let $P = MN$ be the Levi decomposition of the Siegel parabolic subgroup P, where

$$P = \mathrm{GSp}(4,F) \cap \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & * & * \end{bmatrix}, \qquad N = \{\begin{bmatrix} 1 & & x & y \\ & 1 & y & z \\ & & 1 & \\ & & & 1 \end{bmatrix} :\ x,y,z \in F\} \tag{2.1.2}$$

and $M = \{\left[\begin{smallmatrix} xA & \\ & {}^tA^{-1} \end{smallmatrix}\right] :\ A \in \mathrm{GL}(2,F),\ x \in F^\times\}$. We let

$$H := \{\left[\begin{smallmatrix} xI_2 & \\ & I_2 \end{smallmatrix}\right] : x \in F^\times\} \cong F^\times. \tag{2.1.3}$$

Let

$$\beta = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}, \qquad a,b,c \in F \tag{2.1.4}$$

be a symmetric matrix. Then β determines a character ψ_β of N by

$$\psi_\beta(\left[\begin{smallmatrix} 1 & X \\ & 1 \end{smallmatrix}\right]) = \psi(\mathrm{tr}(\beta X)), \qquad X = \left[\begin{smallmatrix} x & y \\ y & z \end{smallmatrix}\right]. \tag{2.1.5}$$

Every character of N is of this form for a uniquely determined β. We say that ψ_β is *non-degenerate* if $\beta \in \mathrm{GL}(2,F)$.

Attached to a non-degenerate ψ_β is a quadratic extension L/F. If $-\det(\beta) \notin F^{\times 2}$, we set $L = F(\sqrt{-\det(\beta)})$; this is the *non-split case*. If $-\det(\beta) \in F^{\times 2}$, we set $L = F \oplus F$; this is the *split case*. Let

$$\begin{aligned} A_\beta &= \{g \in M_2(F) :\ {}^tg\beta g = \det(g)\beta\} \\ &= \{\begin{bmatrix} x+yb/2 & yc \\ -ya & x-yb/2 \end{bmatrix} :\ x,y \in F\}. \end{aligned} \tag{2.1.6}$$

Then A_β is an F-algebra isomorphic to L via the map

$$\begin{bmatrix} x+yb/2 & yc \\ -ya & x-yb/2 \end{bmatrix} \longmapsto x + y\Delta, \tag{2.1.7}$$

where $\Delta = \sqrt{-\det(\beta)}$ in the non-split case, and $\Delta = (-\delta, \delta)$ if $-\det(\beta) = \delta^2$ in the split case.

Let T be the connected component of the stabilizer of ψ_β in M. It is easy to check that $T \cong A_\beta^\times \cong L^\times$. We always consider T a subgroup of $\mathrm{GSp}(4, F)$ via

$$T \ni g \longmapsto \begin{bmatrix} g & \\ & \det(g)\,{}^t g^{-1} \end{bmatrix}. \tag{2.1.8}$$

Explicitly, T consists of all elements

$$\begin{bmatrix} x+y\frac{b}{2} & yc & & \\ -ya & x-y\frac{b}{2} & & \\ & & x-y\frac{b}{2} & ya \\ & & -yc & x+y\frac{b}{2} \end{bmatrix}, \qquad x, y \in F,\ x^2 - y^2\Delta^2 \neq 0. \tag{2.1.9}$$

Let $R := TN$ be the *Bessel subgroup* of $\mathrm{GSp}(4, F)$. If Λ is a character of T, then we can define a character $\Lambda \otimes \psi_\beta$ of R by $tn \mapsto \Lambda(t)\psi_\beta(n)$ for $t \in T$ and $n \in N$.

2.2 Bessel models

Let (π, V) be an irreducible, admissible representation of $\mathrm{GSp}(4, F)$. Non-zero elements of $\mathrm{Hom}_R(V, \mathbb{C}_{\Lambda\otimes\psi_\beta})$ are called (Λ, β)-*Bessel functionals*. It is known that if such a Bessel functional ℓ exists, then $\mathrm{Hom}_R(V, \mathbb{C}_{\Lambda\otimes\psi_\beta})$ is one-dimensional. In this case the space of functions

$$\mathcal{B}(\pi, \Lambda, \beta) := \{B_v : g \mapsto \ell(\pi(v)g) : v \in V\}, \tag{2.2.1}$$

endowed with the action of $\mathrm{GSp}(4, F)$ given by right translations, is called the (Λ, β)-*Bessel model* of π.

Chapter 3

Jacquet-Waldspurger modules

3.1 Jacquet modules

Let (π, V) be an irreducible, admissible representation of $\mathrm{GSp}(4, F)$,

$$V(N) = \langle \pi(n)v - v \mid v \in V,\ n \in N \rangle \qquad \text{and} \qquad V_N = V/V(N)$$

be the usual Jacquet module with respect to the Siegel parabolic subgroup. We identify M with $\mathrm{GL}(2, F) \times \mathrm{GL}(1, F)$ via the map

$$(A, x) \longmapsto \begin{bmatrix} xA & \\ & \det(A)\,{}^tA^{-1} \end{bmatrix}, \qquad A \in \mathrm{GL}(2, F),\ x \in F^\times. \tag{3.1.1}$$

so V_N carries an action of M, and thus an action of $\mathrm{GL}(2, F) \times \mathrm{GL}(1, F)$ via this isomorphism. We have tabulated the semisimplifications of these Jacquet modules in Table 3.1. Note that this table differs from Table A.3 of [10] in three ways:

- In [10] a different version of $\mathrm{GSp}(4, F)$ was used. Switching the last two rows and columns provides an isomorphism.

- The Jacquet modules listed in Table A.3 of [10] are normalized, while the Jacquet modules listed in Table 3.1 are not. The normalized Jacquet module is obtained from the unnormalized one by twisting by $\delta_P^{-1/2}$, where

$$\delta_P(\left[\begin{smallmatrix} A & \\ & x\,{}^tA^{-1} \end{smallmatrix}\right]) = |x^{-1}\det(A)|^3.$$

Hence, we replace each component $\tau \otimes \sigma$ in Table A.3 of [10] by $(\nu^{3/2}\tau) \otimes (\nu^{-3/2}\sigma)$ in order to obtain the unnormalized Jacquet modules.

- In [10] the isomorphism

$$(A, x) \longmapsto \left[\begin{smallmatrix} A & \\ & x\,{}^tA^{-1} \end{smallmatrix}\right], \qquad A \in \mathrm{GL}(2, F),\ x \in F^\times, \tag{3.1.2}$$

was used. Calculations show that we have to replace each component $(\nu^{3/2}\tau) \otimes (\nu^{-3/2}\sigma)$ of the unnormalized Jacquet module by $(\sigma\tau) \otimes (\nu^{3/2}\omega_\tau\sigma)$.

3.2 Waldspurger functionals for GL(2)

Recall the algebra $A_\beta \subset M_2(F)$ defined in (2.1.6), and its unit group $T \subset \mathrm{GL}(2, F)$. Let Λ be a character of T. Let (τ, V) be a smooth representation of $\mathrm{GL}(2, F)$ admitting a central character ω_τ. A Λ-*Waldspurger functional* on τ is a non-zero linear map $\delta : V \to \mathbb{C}$ such that

$$\delta(\tau(t)v) = \Lambda(t)\delta(v) \qquad \text{for all } v \in V \text{ and } t \in T.$$

Since T contains the center Z of $\mathrm{GL}(2, F)$, a necessary condition for such a δ to exist is that $\Lambda|_{F^\times} = \omega_\tau$. As in the case of Bessel functionals, we call a Waldspurger

Table 3.1: Jacquet modules with respect to P, using the isomorphism (3.1.1).

		representation	semisimplification
I		$\chi_1 \times \chi_2 \rtimes \sigma$ (irreducible)	$\sigma(\chi_1 \times \chi_2) \otimes \nu^{3/2}\chi_1\chi_2\sigma$ $+\sigma(\chi_2 \times \chi_1) \otimes \nu^{3/2}\sigma$ $+\sigma(\chi_1\chi_2 \times 1_{F^\times}) \otimes \nu^{3/2}\chi_1\sigma$ $+\sigma(\chi_1\chi_2 \times 1_{F^\times}) \otimes \nu^{3/2}\chi_2\sigma$
II	a	$\chi\mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$	$\sigma\chi\mathrm{St}_{\mathrm{GL}(2)} \otimes \nu^{3/2}\chi^2\sigma$ $+\sigma\chi\mathrm{St}_{\mathrm{GL}(2)} \otimes \nu^{3/2}\sigma$ $+(\chi^2\sigma \times \sigma) \otimes \nu^2\chi\sigma$
	b	$\chi 1_{\mathrm{GL}(2)} \rtimes \sigma$	$\sigma\chi 1_{\mathrm{GL}(2)} \otimes \nu^{3/2}\chi^2\sigma$ $+\sigma\chi 1_{\mathrm{GL}(2)} \otimes \nu^{3/2}\sigma$ $+(\chi^2\sigma \times \sigma) \otimes \nu\chi\sigma$
III	a	$\chi \rtimes \sigma\mathrm{St}_{\mathrm{GSp}(2)}$	$\sigma(\chi\nu^{-1/2} \times \nu^{1/2}) \otimes \chi\nu^2\sigma$ $+\sigma(\chi\nu^{1/2} \times \nu^{-1/2}) \otimes \nu^2\sigma$
	b	$\chi \rtimes \sigma 1_{\mathrm{GSp}(2)}$	$\sigma(\chi\nu^{1/2} \times \nu^{-1/2}) \otimes \chi\nu\sigma$ $+\sigma(\chi\nu^{-1/2} \times \nu^{1/2}) \otimes \nu\sigma$
IV	a	$\sigma\mathrm{St}_{\mathrm{GSp}(4)}$	$\sigma\mathrm{St}_{\mathrm{GL}(2)} \otimes \nu^3\sigma$
	b	$L(\nu^2, \nu^{-1}\sigma\mathrm{St}_{\mathrm{GSp}(2)})$	$\sigma 1_{\mathrm{GL}(2)} \otimes \nu^3\sigma$ $+\sigma(\nu^{3/2} \times \nu^{-3/2}) \otimes \nu\sigma$
	c	$L(\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-3/2}\sigma)$	$\sigma\mathrm{St}_{\mathrm{GL}(2)} \otimes \sigma$ $+\sigma(\nu^{3/2} \times \nu^{-3/2}) \otimes \nu^2\sigma$
	d	$\sigma 1_{\mathrm{GSp}(4)}$	$\sigma 1_{\mathrm{GL}(2)} \otimes \sigma$
V	a	$\delta([\xi, \nu\xi], \nu^{-1/2}\sigma)$	$\sigma\xi\mathrm{St}_{\mathrm{GL}(2)} \otimes \nu^2\sigma + \sigma\mathrm{St}_{\mathrm{GL}(2)} \otimes \xi\nu^2\sigma$
	b	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$	$\sigma\xi\mathrm{St}_{\mathrm{GL}(2)} \otimes \nu\sigma + \sigma 1_{\mathrm{GL}(2)} \otimes \xi\nu^2\sigma$
	c	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)}, \xi\nu^{-1/2}\sigma)$	$\sigma\mathrm{St}_{\mathrm{GL}(2)} \otimes \xi\nu\sigma + \sigma\xi 1_{\mathrm{GL}(2)} \otimes \nu^2\sigma$
	d	$L(\nu\xi, \xi \rtimes \nu^{-1/2}\sigma)$	$\sigma 1_{\mathrm{GL}(2)} \otimes \xi\nu\sigma + \sigma\xi 1_{\mathrm{GL}(2)} \otimes \nu\sigma$

		representation	semisimplification
VI	a	$\tau(S,\nu^{-1/2}\sigma)$	$2\cdot(\sigma\mathrm{St}_{\mathrm{GL}(2)}\otimes\nu^2\sigma)+\sigma 1_{\mathrm{GL}(2)}\otimes\nu^2\sigma$
	b	$\tau(T,\nu^{-1/2}\sigma)$	$\sigma 1_{\mathrm{GL}(2)}\otimes\nu^2\sigma$
	c	$L(\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma)$	$\sigma\mathrm{St}_{\mathrm{GL}(2)}\otimes\nu\sigma$
	d	$L(\nu,1_{F^\times}\rtimes\nu^{-1/2}\sigma)$	$2\cdot(\sigma 1_{\mathrm{GL}(2)}\otimes\nu\sigma)+\sigma\mathrm{St}_{\mathrm{GL}(2)}\otimes\nu\sigma$
VII		$\chi\rtimes\pi$	0
VIII	a	$\tau(S,\pi)$	0
	b	$\tau(T,\pi)$	0
IX	a	$\delta(\nu\xi,\nu^{-1/2}\pi(\mu))$	0
	b	$L(\nu\xi,\nu^{-1/2}\pi(\mu))$	0
X		$\pi\rtimes\sigma$	$\sigma\pi\otimes\nu^{3/2}\omega_\pi\sigma+\sigma\pi\otimes\nu^{3/2}\sigma$
XI	a	$\delta(\nu^{1/2}\pi,\nu^{-1/2}\sigma)$	$\sigma\pi\otimes\nu^2\sigma$
	b	$L(\nu^{1/2}\pi,\nu^{-1/2}\sigma)$	$\sigma\pi\otimes\nu\sigma$
		supercuspidal	0

functional *split* if $-\det(\beta)\in F^{\times 2}$, otherwise *non-split*.

The (Λ,β)-Waldspurger functionals are the non-zero elements of the space $\mathrm{Hom}_T(\tau,\mathbb{C}_\Lambda)$. If we put

$$V(T,\Lambda)=\langle\tau(t)v-\Lambda(t)v\ :\ v\in V,\ t\in T\rangle\qquad\text{and}\qquad V_{T,\Lambda}=V/V(T,\Lambda),\tag{3.2.1}$$

then $\mathrm{Hom}_T(\tau,\mathbb{C}_\Lambda)\cong\mathrm{Hom}(V_{T,\Lambda},\mathbb{C})$. Note that if L is a field, so that T/Z is compact, then the space $V(T,\Lambda)$ can also be characterized as follows,

$$V(T,\Lambda)=\Big\{v\in V\ :\ \int_{T/Z}\Lambda(t)^{-1}\tau(t)v\,dt=0\Big\}.\tag{3.2.2}$$

The map $V\mapsto V_{T,\Lambda}$ defines a functor, called the *Waldspurger functor*, from the category of smooth representations of $\mathrm{GL}(2,F)$ to the category of $F^\times$-modules. This can be seen just as the analogous statement in the case of Jacquet modules. In particular, if L is a field, then the Waldspurger functor is exact; this follows from (3.2.2) with similar arguments as in Proposition 2.35 of [2].

Now assume that (τ, V) is irreducible and admissible. Then it is known by [16], [12] and Lemma 8 of [17] that the space $\mathrm{Hom}_T(\tau, \mathbb{C}_\Lambda)$ is at most one-dimensional. It follows that

$$\dim V_{T,\Lambda} \leq 1. \tag{3.2.3}$$

The following facts are known for any character Λ of T such that $\Lambda|_{F^\times} = \omega_\tau$:

- For principal series representations, we have

$$\dim(\mathrm{Hom}_T(\chi_1 \times \chi_2, \mathbb{C}_\Lambda)) = 1 \qquad \text{for all } \Lambda; \tag{3.2.4}$$

  see Proposition 1.6 and Theorem 2.3 of [16].

- For twists of the Steinberg representation, we have

$$\dim(\mathrm{Hom}_T(\sigma\mathrm{St}_{\mathrm{GL}(2)}, \mathbb{C}_\Lambda)) = \begin{cases} 0 & \text{if } L \text{ is a field and } \Lambda = \sigma \circ \mathrm{N}_{L/F}, \\ 1 & \text{otherwise}; \end{cases} \tag{3.2.5}$$

  see Proposition 1.7 and Theorem 2.4 of [16].

- If τ is infinite-dimensional and $L = F \times F$, then

$$\dim(\mathrm{Hom}_T(\pi, \mathbb{C}_\Lambda)) = 1 \qquad \text{for all } \Lambda; \tag{3.2.6}$$

  see Lemme 8 of [17].

- For one-dimensional representations, we have

$$\dim(\mathrm{Hom}_T(\sigma 1_{\mathrm{GL}(2)}, \mathbb{C}_\Lambda)) = \begin{cases} 1 & \text{if } \Lambda = \sigma \circ \mathrm{N}_{L/F}, \\ 0 & \text{otherwise;} \end{cases} \tag{3.2.7}$$

  this is obvious.

3.2.1 Finite-dimensional $F^\times$-modules

Recall that $F^\times = \langle \varpi \rangle \times \mathfrak{o}^\times$. We consider representations of $F^\times$ on finite-dimensional complex vector spaces. All such are assumed to be continuous.

Let n be a positive integer and U be an n-dimensional complex vector space with basis $e_1, \ldots, e_n$. We define an action of $F^\times$ on U as follows:

- $\mathfrak{o}^\times$ acts trivially on all of U.
- ϖ acts by sending e_j to $e_j + e_{j-1}$ for all $j \in \{1, \ldots, n\}$, where we understand $e_0 = 0$. In other words, the matrix of ϖ with respect to the basis $e_1, \ldots, e_n$ is a Jordan block

$$\begin{bmatrix} 1 & 1 & & \\ & \ddots & \ddots & \\ & & 1 & 1 \\ & & & 1 \end{bmatrix}. \tag{3.2.8}$$

We denote the equivalence class of the $F^\times$-module thus defined by $[n]$. Note that $[n]$ is canonically defined, even though ϖ is not. Clearly, $[n]$ is an indecomposable $F^\times$-module. If σ is a character of $F^\times$, then $\sigma[n] := \sigma \otimes [n]$ is also indecomposable.

Lemma 3.2.1. *Every finite-dimensional indecomposable $F^\times$-module is of the form $\sigma[n]$ for some character σ of $F^\times$ and positive integer n.*

Proof. Let (φ, U) be an indecomposable $F^\times$-module. We may decompose U over

$\mathfrak{o}^\times$, i.e.,

$$U = \bigoplus_{i=1}^{r} U(\sigma_i), \tag{3.2.9}$$

where σ_i are pairwise distinct characters of $F^\times$, and

$$U(\sigma_i) = \{u \in U \,:\, \varphi(x)u = \sigma_i(x)u \text{ for all } x \in \mathfrak{o}^\times\}. \tag{3.2.10}$$

Let $f = \varphi(\varpi)$. Since each $U(\sigma_i)$ is f-invariant and U is indecomposable, it follows that $r = 1$, i.e., $U = U(\sigma)$ for some character σ of $\mathfrak{o}^\times$. Indecomposability implies that the Jordan normal form of f consists of only one Jordan block

$$\begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \lambda & 1 \\ & & & \lambda \end{bmatrix}, \qquad \lambda \in \mathbb{C}^\times, \tag{3.2.11}$$

of size n. Extend σ to a character of $F^\times$ by setting $\sigma(\varpi) = \lambda$. Then it is easy to see that $\varphi \cong \sigma[n]$. $\square$

Lemma 3.2.2. *Let U be a finite-dimensional $F^\times$-module. Then*

$$U \cong \bigoplus_{i=1}^{r} \sigma_i[n_i] \tag{3.2.12}$$

with characters σ_i of $F^\times$ and positive integers n_i. A decomposition as in (3.2.12) *is unique up to permutation of the summands.*

Proof. A decomposition as in (3.2.12) exists by Lemma 3.2.1. To prove uniqueness, assume that

$$\bigoplus_{i=1}^{r} \sigma_i[n_i] \cong \bigoplus_{j=1}^{s} \tau_j[n_j]. \tag{3.2.13}$$

By considering isotypical components with respect to characters of $\mathfrak{o}^\times$, we may assume that all σ_i and τ_j agree when restricted to $\mathfrak{o}^\times$. After appropriate tensoring we may assume this restriction is trivial. The uniqueness statement then follows from the uniqueness of Jordan normal forms. $\square$

Lemma 3.2.3. *Let σ be a character of $F^\times$, and n a positive integer. Let $m \in \{0, \dots, n\}$.*

i) There exists exactly one $F^\times$-invariant submodule U_m of $\sigma[n]$ of dimension m. We have $U_k \subset U_m$ for $k \leq m$.

ii) The representation of $F^\times$ on U_m is isomorphic to $\sigma[m]$.

iii) The representation of $F^\times$ on $\sigma[n]/U_m$ is isomorphic to $\sigma[n-m]$.

Proof. i) Since the invariant subspaces of $[n]$ and $\sigma[n]$ coincide, we may assume that $\sigma = 1$, so that $\sigma[n] = [n]$. Let $e_1, \dots, e_n$ be a basis of $[n]$ with respect to which ϖ acts via the matrix (3.2.8). Let $U_m = \langle e_1, \dots, e_m \rangle$. Then U_m is invariant and isomorphic to $[m]$ as an $F^\times$-module.

Conversely, let $U \subset [n]$ be any non-zero invariant subspace. Then U is also invariant under the endomorphism f with matrix

$$\begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}. \tag{3.2.14}$$

The effect of f on a column vector u is to shift its entries "up" and fill in a 0 at the bottom. Let m be maximal with the property that there exists a $u \in U$ of the form

$$u = {}^t[u_1, \dots, u_m, 0, \dots, 0] \qquad \text{with } u_m \neq 0.$$

The vector $f^{m-1}u$ is a non-zero multiple of e_1, showing that $e_1 \in U$. Considering

$f^{m-2}u$, we see that $e_2 \in U$ as well. Continuing, we see that $e_1, \ldots, e_m \in U$. The maximality of m implies that $U = U_m$.

ii) We already saw that the subspace U_m of $[n]$ is isomorphic to $[m]$. Hence the subspace $\sigma \otimes U_m$ of $\sigma[n]$ is isomorphic to $\sigma[m]$.

iii) Clearly $[n]/U_m$ is isomorphic to $[n-m]$. Hence $\sigma[n]/(\sigma \otimes U_m)$ is isomorphic to $\sigma[n-m]$. □

Let U be a finite-dimensional $F^\times$-module. For a character σ of $F^\times$, let U_σ be the sum of all submodules of U isomorphic to $\sigma[n]$ for some n. We call U_σ the *σ-component* of U. By (3.2.12), U is the direct sum of its σ-components. A homomorphism $U \to V$ of finite-dimensional $F^\times$-modules induces a map $U_\sigma \to V_\sigma$ for all σ; this follows from Lemma 3.2.3.

3.2.2 Split Waldspurger functors

In this section, we consider the case of split Waldspurger models with $\beta = \begin{bmatrix} & 1/2 \\ 1/2 & \end{bmatrix}$ and of split Waldspurger functionals as

$$T = \begin{bmatrix} * & \\ & * \end{bmatrix} \cap \mathrm{GL}(2, F) \cong F^\times \times F^\times. \tag{3.2.15}$$

Let Λ be a character of T. In fact, Λ can be written as a product of characters Λ_1 and Λ_2 of $F^\times$ as follows:

$$\Lambda(\begin{bmatrix} a & \\ & b \end{bmatrix}) = \Lambda_1(a)\Lambda_2(b). \tag{3.2.16}$$

We denote $\Lambda = (\Lambda_1, \Lambda_2)$. It is straightforward to show that

$$V(T, \Lambda) = \langle \pi(\left[\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right])v - \Lambda_1(a)v \,|\, v \in V,\ a \in F^\times \rangle. \tag{3.2.17}$$

We will give an alternative description of this space. Write $a \in F^\times$ as $a = \varpi^n u$ with $n \in \mathbb{Z}$ and $u \in \mathfrak{o}^\times$. Then

$$\pi(\left[\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right])v - \Lambda_1(a)v = \pi(\left[\begin{smallmatrix} \varpi^n & \\ & 1 \end{smallmatrix}\right])v_1 - \Lambda_1(\varpi^n)v_1 + \pi(\left[\begin{smallmatrix} u & \\ & 1 \end{smallmatrix}\right])v_2 - \Lambda_1(u)v_2, \tag{3.2.18}$$

where

$$v_1 = \pi(\left[\begin{smallmatrix} u & \\ & 1 \end{smallmatrix}\right])v, \qquad v_2 = \Lambda_1(\varpi)^n v.$$

This shows that $V(T, \Lambda) = V(T, \Lambda)' + V(T, \Lambda)''$, where

$$V(T, \Lambda)' = \langle \pi(\left[\begin{smallmatrix} \varpi^n & \\ & 1 \end{smallmatrix}\right])v - \Lambda_1(\varpi^n)v :\ v \in V,\ n \in \mathbb{Z} \rangle \tag{3.2.19}$$

and

$$V(T, \Lambda)'' = \langle \pi(\left[\begin{smallmatrix} u & \\ & 1 \end{smallmatrix}\right])v - \Lambda_1(u)v \,|\, v \in V,\ u \in \mathfrak{o}^\times \rangle. \tag{3.2.20}$$

Since $\mathfrak{o}^\times$ is compact, the space $V(T, \Lambda)''$ can be characterized as

$$V(T, \Lambda)'' = \Big\{ v \in V \ \Big| \ \int_{\mathfrak{o}^\times} \Lambda_1(u)^{-1} \pi(\left[\begin{smallmatrix} u & \\ & 1 \end{smallmatrix}\right])v \, du = 0 \Big\}. \tag{3.2.21}$$

Since

$$\pi(\left[\begin{smallmatrix} \varpi^n & \\ & 1 \end{smallmatrix}\right])v - \Lambda_1(\varpi^n)v = \sum_{i=0}^{n-1} \Big(\pi(\left[\begin{smallmatrix} \varpi & \\ & 1 \end{smallmatrix}\right])v_i - \Lambda_1(\varpi)v_i \Big), \quad v_i = \Lambda_1(\varpi^i)\pi(\left[\begin{smallmatrix} \varpi^{n-1-i} & \\ & 1 \end{smallmatrix}\right])v, \tag{3.2.22}$$

we have

$$\begin{aligned} V(T,\Lambda)' &= \langle \pi([\begin{smallmatrix} \varpi & \\ & 1 \end{smallmatrix}])v - \Lambda_1(\varpi)v \mid v \in V \rangle \\ &= \Big(\pi([\begin{smallmatrix} \varpi & \\ & 1 \end{smallmatrix}]) - \Lambda_1(\varpi)\mathrm{id}_V\Big)V. \end{aligned} \tag{3.2.23}$$

We summarize the discussion: If $v \in V(T,\Lambda)$, then there exist $v_1, v_2 \in V$ such that

$$v = \pi([\begin{smallmatrix} \varpi & \\ & 1 \end{smallmatrix}])v_1 - \Lambda_1(\varpi)v_1 + v_2, \qquad \text{where} \int\limits_{\mathfrak{o}^\times} \Lambda_1(u)^{-1}\pi([\begin{smallmatrix} u & \\ & 1 \end{smallmatrix}])v_2\, du = 0. \tag{3.2.24}$$

Recall that every irreducible, admissible, infinite-dimensional representation (π, V) of $\mathrm{GL}(2,F)$ admits a split (T,Λ)-Waldspurger functional, unique up to scalars, for any Λ. Such a functional can be constructed by means of zeta integrals, as follows. First, choose any unitary character σ of $F^\times$ such that $\sigma|_{\mathfrak{o}^\times} = \Lambda_1^{-1}|_{\mathfrak{o}^\times}$. For W in a ψ-Whittaker model of π, let

$$Z_\sigma(s,W) = \int\limits_{F^\times} W([\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}])|a|^{s-1/2}\sigma(a)\, d^\times a. \tag{3.2.25}$$

These integrals are convergent for $\mathrm{Re}(s) \gg 0$ and have meromorphic continuation to all of $\mathbb{C}$. In fact, these are the standard zeta integrals for the twisted representation $\sigma\pi$. Clearly, they satisfy

$$Z_\sigma(s, \pi([\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}])W) = |a|^{-s+1/2}\sigma(a)^{-1}Z_\sigma(s,W) \qquad \text{for } a \in F^\times \tag{3.2.26}$$

for all s where $Z_\sigma(s,W)$ does not have a pole (for *any* W). Choose s_0 such that $|\cdot|^{-s_0+1/2}\sigma^{-1} = \Lambda_1$ and assume that s_0 is not a pole of $Z_\sigma(s,W)$. Then

$$Z_\sigma(s_0, \pi([{}^{a}{}_{1}])W) = \Lambda_1(a) Z_\sigma(s_0, W) \qquad \text{for } a \in F^\times, \tag{3.2.27}$$

and hence the map $W \mapsto Z_\sigma(s_0, W)$ is a (T,Λ)-Waldspurger functional.

Assume however that s_0 is a pole of $Z_\sigma(s,W)$ (for *some* W). This is equivalent to saying that s_0 is a pole of $L(s,\sigma\pi)$. By definition of the L-factor, the quotient

$$\frac{Z_\sigma(s,W)}{L(s,\sigma\pi)}$$

has analytic continuation to an *entire* function. In fact, this quotient lies in $\mathbb{C}[q^s, q^{-s}]$. We can therefore consider the functional

$$W \longmapsto \frac{L_\sigma(s_0, W)}{L(s_0, \sigma\pi)}. \tag{3.2.28}$$

This functional again is a (T,Λ)-Waldspurger functional.

Now assume that $\pi = \chi_1 \times \chi_2$ is an irreducible principal series representation. Recall from above that the Jacquet module is $V_N = \chi_1|\cdot|^{1/2} + \chi_2|\cdot|^{1/2}$ as a representation of $[{}^{*}{}_{1}] \cong F^\times$. As above, choose s_0 such that $|\cdot|^{-s_0+1/2}\sigma^{-1} = \Lambda_1$. It is easy to see that the following are equivalent:

i) $Z_\sigma(s,W)$ has no pole at $s = s_0$, for any W.

ii) $L(s,\sigma\pi)$ has no pole at $s = s_0$.

iii) $\Lambda_1 \neq \chi_1|\cdot|^{1/2}$ and $\Lambda_1 \neq \chi_2|\cdot|^{1/2}$.

iv) Λ_1 is not a subquotient of V_N.

If these conditions are satisfied, then the map $W \mapsto Z_\sigma(s_0, W)$ is a (T,Λ)-Waldspurger functional. In any case, the map (3.2.28) is a (T,Λ)-Waldspurger functional.

Assume that $\Lambda_1 = \chi_1|\cdot|^{1/2}$, so that the above conditions are not satisfied. Then we can alternatively construct a (T,Λ)-Waldspurger functional in the following ways:

- As the composition

$$V \longrightarrow V_N \longrightarrow V_N \Big/ \Big\langle \left[\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right] v - \chi_1(a)|a|^{1/2} v \Big\rangle \xrightarrow{\sim} \mathbb{C}. \tag{3.2.29}$$

  (This works analogously for $\Lambda_1 = \chi_2|\cdot|^{1/2}$.)

- As the map $f \mapsto f(1)$ in the standard induced model. (For $\Lambda_1 = \chi_2|\cdot|^{1/2}$ and $\chi_2 \neq \chi_1$, we can take the map $f \mapsto (Mf)(1)$, where M is the standard intertwining operator.)

Exactness properties of the functor $V \mapsto V_{T,\Lambda}$

The functor $V \mapsto V_{T,\Lambda}$ from smooth $\mathrm{GL}(2,F)$-modules to vector spaces is not exact, but still has the following exactness properties:

- If $0 \to V' \to V \to V'' \to 0$ is exact, then

$$V'_{T,\Lambda} \longrightarrow V_{T,\Lambda} \longrightarrow V''_{T,\Lambda} \longrightarrow 0 \tag{3.2.30}$$

  is exact; see Proposition 2.35 of [2].

- If $V = V' \oplus V''$ is a direct sum, then

$$V_{T,\Lambda} = V'_{T,\Lambda} \oplus V''_{T,\Lambda}; \tag{3.2.31}$$

  this is easy to see.

In this subsection we will prove additional exactness properties on a subcategory of the category of all smooth $\mathrm{GL}(2, F)$-modules. This subcategory, which we call $\mathcal{C}$, consists of all smooth $\mathrm{GL}(2, F)$-modules V that admit a filtration

$$0 = V_0 \subset V_1 \subset \ldots \subset V_n = V \tag{3.2.32}$$

such that each V_i/V_{i-1} is irreducible, admissible and infinite-dimensional. We will use the fact that the objects (τ, V) in $\mathcal{C}$ have the following property:

$$\text{If } v \in V \text{ satisfies } \tau(\left[\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right])v = \lambda(a)v \text{ for all } a \in F^\times, \text{ then } v = 0. \tag{3.2.33}$$

Here, λ is any character of $F^\times$. To see why (3.2.33) is true, note that v induces a vector φ with the analogous property in some irreducible subquotient of V. Considering φ in the Kirillov model of this irreducible subquotient, we can evaluate at $x \in F^\times$ to obtain $\varphi(ax) = \lambda(a)\varphi(x)$ for all $a, x \in F^\times$. Since $\varphi(x) = 0$ for $v(x) \ll 0$ (which is a property shared by all functions in the Kirillov model), it follows that $\varphi(x) = 0$ for all $x \in F^\times$.

Lemma 3.2.4. *Let (π, V) be a smooth representation of $\mathrm{GL}(2, F)$. Let $0 \subset V_1 \subset V$ be a filtration of $\mathrm{GL}(2, F)$-modules such that the representation τ on V/V_1 has* *the following property:*

$$\text{If } W \in V/V_1 \text{ satisfies } \tau(\left[\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right])W = \Lambda_1(a)W \text{ for all } a \in F^\times, \text{ then } W = 0. \tag{3.2.34}$$

(In particular, this is satisfied if V/V_1 is in $\mathcal{C}$.) Let U be the subspace of V_1 of vectors v_1 satisfying

$$\pi(\left[\begin{smallmatrix} u & \\ & 1 \end{smallmatrix}\right])v_1 = \Lambda_1(u)v_1 \qquad \text{for all } u \in \mathfrak{o}^\times. \tag{3.2.35}$$

Then $U \cap V(T, \Lambda) = U \cap V_1(T, \Lambda)$.

Proof. Assume that $v_1 \in U \cap V(T, \Lambda)$. By (3.2.24), we can write

$$v_1 = \pi(\left[\begin{smallmatrix} \varpi & \\ & 1 \end{smallmatrix}\right])v_2 - \Lambda_1(\varpi)v_2 + v_3, \qquad \text{where} \int\limits_{\mathfrak{o}^\times} \Lambda_1(u)^{-1}\pi(\left[\begin{smallmatrix} u & \\ & 1 \end{smallmatrix}\right])v_3 \, du = 0.$$

Applying

$$\frac{1}{\mathrm{vol}(\mathfrak{o}^\times)} \int\limits_{\mathfrak{o}^\times} \Lambda_1(u)^{-1}\pi(\left[\begin{smallmatrix} u & \\ & 1 \end{smallmatrix}\right])(\ldots) \, du$$

to both sides and observing (3.2.35), we get

$$v_1 = \pi(\left[\begin{smallmatrix} \varpi & \\ & 1 \end{smallmatrix}\right])v_2' - \Lambda_1(\varpi)v_2', \qquad \text{where } v_2' = \int\limits_{\mathfrak{o}^\times} \Lambda_1(u)^{-1}\pi(\left[\begin{smallmatrix} u & \\ & 1 \end{smallmatrix}\right])v_2 \, du. \tag{3.2.36}$$

Clearly, the vector v_2' satisfies

$$\pi(\left[\begin{smallmatrix} u & \\ & 1 \end{smallmatrix}\right])v_2' = \Lambda_1(u)v_2' \qquad \text{for all } u \in \mathfrak{o}^\times. \tag{3.2.37}$$

Let τ be the representation of $\mathrm{GL}(2, F)$ on V/V_1. Applying the projection $V \to$

V/V_1 to both sides of the equation (3.2.36), we obtain

$$\tau(\left[\begin{smallmatrix}\varpi & \\ & 1\end{smallmatrix}\right])W = \Lambda_1(\varpi)W, \tag{3.2.38}$$

where W is the image of v_2' in V/V_1. By (3.2.37),

$$\tau(\left[\begin{smallmatrix}u & \\ & 1\end{smallmatrix}\right])W = \Lambda_1(u)W \qquad \text{for all } u \in \mathfrak{o}^\times. \tag{3.2.39}$$

Combining (3.2.38) and (3.2.39), we see that

$$\tau(\left[\begin{smallmatrix}a & \\ & 1\end{smallmatrix}\right])W = \Lambda_1(a)W \qquad \text{for all } a \in F^\times. \tag{3.2.40}$$

By hypothesis (3.2.34), it follows that $W = 0$. Hence $v_2' \in V_1$. Then $v_1 \in V_1(T, \Lambda)$ by (3.2.36), concluding the proof. $\square$

Lemma 3.2.5. *Let (π, V) be an irreducible, admissible, infinite-dimensional representation of* $\mathrm{GL}(2, F)$*. Let U be the subspace of vectors $v \in V$ satisfying*

$$\pi(\left[\begin{smallmatrix}u & \\ & 1\end{smallmatrix}\right])v = \Lambda_1(u)v \qquad \textit{for all } u \in \mathfrak{o}^\times. \tag{3.2.41}$$

Then $U/(U \cap V(T, \Lambda)$ is one-dimensional.

Proof. Since $U/(U \cap V(T, \Lambda)$ injects into $V/V(T, \Lambda)$, and since $V/V(T, \Lambda)$ is one-dimensional by the existence and uniqueness of split Waldspurger functionals, we only need to show that U is not contained in $V(T, \Lambda)$.

We may assume that V is the Kirillov model of π. Recall that the action of T

on V is determined by

$$(\pi([\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}])\varphi)(y) = \varphi(ay), \qquad a, y \in F^\times.$$

Recall also that V contains $\mathcal{S}(F^\times)$ as a subspace of codimension 0, 1 or 2. We have $\mathcal{S}(F^\times) = V$ if and only if π is supercuspidal. We are in the codimension 1 case if and only if $\pi = \chi \mathrm{St}_{\mathrm{GL}(2)}$; in this case the asymptotic behavior of the functions $\varphi \in V$ is given by

$$\varphi(a) = C|a|\chi(a) \qquad \text{for } |a| < \varepsilon. \tag{3.2.42}$$

We are in the codimension 2 case if and only if $\pi = \chi_1 \times \chi_2$ is a principal series representation; in this case the asymptotic behavior of the functions $\varphi \in V$ is given by

$$\varphi(a) = C_1|a|^{1/2}\chi_1(a) + C_2|a|^{1/2}\chi_2(a) \qquad \text{for } |a| < \varepsilon \tag{3.2.43}$$

if $\chi_1 \neq \chi_2$, and by

$$\varphi(a) = (C_1 + C_2 v(a))|a|^{1/2}\chi_1(a) \qquad \text{for } |a| < \varepsilon \tag{3.2.44}$$

if $\chi_1 = \chi_2$.

We now distinguish two cases. Assume first that Λ_1 is not equal to $|\cdot|\chi$ in case (3.2.42), not equal to $|\cdot|^{1/2}\chi_1$ or $|\cdot|^{1/2}\chi_2$ in case (3.2.43), and not equal to $|\cdot|^{1/2}\chi_1$ in case (3.2.44) (in other words, Λ_1 is not one of the characters occurring in the Jacquet module V_N). Consider the function $\varphi_0 \in V$ given by

$$\varphi_0(u) = \begin{cases} \Lambda_1(u) & \text{if } u \in \mathfrak{o}^\times, \\ 0 & \text{if } u \notin \mathfrak{o}^\times. \end{cases} \tag{3.2.45}$$

Clearly, $\varphi_0 \in U$. Assume that $\varphi_0 \in V(T, \Lambda)$; we will obtain a contradiction. By (3.2.24), we can write

$$\varphi_0 = \pi([\begin{smallmatrix} \varpi & \\ & 1 \end{smallmatrix}])\varphi_1 - \Lambda_1(\varpi)\varphi_1 + \varphi_2, \qquad \text{where} \int\limits_{\mathfrak{o}^\times} \Lambda_1(u)^{-1}\pi([\begin{smallmatrix} u & \\ & 1 \end{smallmatrix}])\varphi_2 \, du = 0. \tag{3.2.46}$$

Applying

$$\frac{1}{\mathrm{vol}(\mathfrak{o}^\times)} \int\limits_{\mathfrak{o}^\times} \Lambda_1(u)^{-1}\pi([\begin{smallmatrix} u & \\ & 1 \end{smallmatrix}])(\ldots) \, du$$

to both sides, we get

$$\varphi_0 = \pi([\begin{smallmatrix} \varpi & \\ & 1 \end{smallmatrix}])\varphi_1' - \Lambda_1(\varpi)\varphi_1', \qquad \text{where } \varphi_1' = \int\limits_{\mathfrak{o}^\times} \Lambda_1(u)^{-1}\pi([\begin{smallmatrix} u & \\ & 1 \end{smallmatrix}])\varphi_1 \, du. \tag{3.2.47}$$

Evaluating at $u\varpi^{-n}$ with $u \in \mathfrak{o}^\times$ and $n > 0$, we get

$$\varphi_1'(\varpi^{-n+1}u) = \Lambda_1(\varpi)\varphi_1'(\varpi^{-n}u) \qquad \text{for } n > 0. \tag{3.2.48}$$

Since $\varphi_1'(a) = 0$ for $v(a) \ll 0$, it follows that $\varphi_1'(u) = 0$ for $u \in \mathfrak{o}^\times$. Further evaluating (3.2.47) at $u\varpi^n$ with $u \in \mathfrak{o}^\times$ and $n \geq 0$, we conclude

$$\varphi_1'(a) = \Lambda_1(\varpi)^{-1}\Lambda_1(a) \qquad \text{for } v(a) \geq 1. \tag{3.2.49}$$

Hence φ_1 has an asymptotic behavior which is not permitted by our assumption on Λ_1. This contradiction proves $U \not\subset V(T,\Lambda)$ under our ssumption.

Now assume that Λ_1 is one of the characters appearing in (3.2.42), (3.2.43) or (3.2.44). Assume first that we are in case (3.2.43) and that $\Lambda_1 = |\cdot|^{1/2}\chi_1$. Define $\varphi_0 \in V$ by

$$\varphi_0(a) = \begin{cases} \Lambda_1(a) & \text{if } v(a) \geq 0, \\ 0 & \text{if } v(a) < 0. \end{cases} \tag{3.2.50}$$

Clearly, $\varphi_0 \in U$. Assume that $\varphi_0 \in V(T,\Lambda)$; we will obtain a contradiction. As above we see that

$$\varphi_0 = \pi(\left[\begin{smallmatrix} \varpi & \\ & 1 \end{smallmatrix}\right])\varphi_1 - \Lambda_1(\varpi)\varphi_1 \tag{3.2.51}$$

for some $\varphi_1 \in V$. Let C_1, C_2 be such that

$$\varphi_1(a) = C_1|a|^{1/2}\chi_1(a) + C_2|a|^{1/2}\chi_2(a) \qquad \text{for } |a| < \varepsilon. \tag{3.2.52}$$

Evaluating (3.2.51) at $a \in F^\times$ with $|a| < \varepsilon$, we see that

$$\Lambda_1(a) = C_2(|\varpi|^{1/2}\chi_2(\varpi) - \Lambda_1(\varpi))|a|^{1/2}\chi_2(a) \qquad \text{for } |a| < \varepsilon. \tag{3.2.53}$$

Since $\Lambda_1 = |\cdot|^{1/2}\chi_1$ and $\chi_1 \neq \chi_2$, this is a contradiction.

Assume next that we are in case (3.2.44) and that $\Lambda_1 = |\cdot|^{1/2}\chi_1$. In this case we define $\varphi_0 \in V$ by

$$\varphi_0(a) = \begin{cases} v(a)\Lambda_1(a) & \text{if } v(a) \geq 0, \\ 0 & \text{if } v(a) < 0. \end{cases} \tag{3.2.54}$$

Then again $\varphi_0 \in U$, and as above we see that $\varphi_0 \notin V(T, \Lambda)$.

The remaining cases are treated similarly, concluding the proof. $\square$

Lemma 3.2.6. *Let*

$$0 \longrightarrow V_1 \longrightarrow V \longrightarrow V_2 \longrightarrow 0 \tag{3.2.55}$$

be an exact sequence such that V_1 is simple (i.e., V_1 is an irreducible, admissible, infinite-dimensional representation of $\mathrm{GL}(2, F)$*), and $0 \subset V_1 \subset V$ satisfies the condition* (3.2.34)*. Then the sequence*

$$0 \longrightarrow (V_1)_{T,\Lambda} \longrightarrow V_{T,\Lambda} \longrightarrow (V_2)_{T,\Lambda} \longrightarrow 0 \tag{3.2.56}$$

is exact.

Proof. We only have to show that $(V_1)_{T,\Lambda} \to V_{T,\Lambda}$ is injective. Let U be the subspace of V_1 of vectors v_1 satisfying (3.2.35). By Lemma 3.2.5, we have $U \cap V_1(T, \Lambda) = U \cap V(T, \Lambda)$. Hence the composition

$$U/(U \cap V_1(T, \Lambda)) \longrightarrow V_1/V_1(T, \Lambda) \longrightarrow V/V(T, \Lambda) \tag{3.2.57}$$

is injective. By Lemma 3.2.5, the space $U/(U \cap V_1(T, \Lambda))$ is one-dimensional. The space $V_1/V_1(T, \Lambda)$ is also one-dimensional by the existence and uniqueness of split Waldspurger functionals. It follows that the first map in (3.2.57) is an isomorphism, and that the second map is injective. $\square$

Proposition 3.2.7. *The functor $V \mapsto V_{T,\Lambda}$ from $\mathcal{C}$ to the category of $\mathbb{C}$-vector spaces is exact.*

Proof. Let

$$0 \longrightarrow V_1 \longrightarrow V \longrightarrow V_2 \longrightarrow 0 \tag{3.2.58}$$

be an exact sequence in $\mathcal{C}$. We will prove by induction on the length of V_1 that

$$0 \longrightarrow (V_1)_{T,\Lambda} \longrightarrow V_{T,\Lambda} \longrightarrow (V_2)_{T,\Lambda} \longrightarrow 0 \tag{3.2.59}$$

is also exact. By (3.2.33) and Lemma 3.2.6, the statement is true if V_1 has length 1. Assume that the length of V_1 is greater than 1. Let V_0 be a simple submodule of V_1. Then we have an exact sequence

$$0 \longrightarrow V_1/V_0 \longrightarrow V/V_0 \longrightarrow V/V_1 \longrightarrow 0. \tag{3.2.60}$$

Since the length of V_1/V_0 is less than the length of V_1, we may assume by induction that the sequence

$$0 \longrightarrow (V_1/V_0)_{T,\Lambda} \longrightarrow (V/V_0)_{T,\Lambda} \longrightarrow (V/V_1)_{T,\Lambda} \longrightarrow 0 \tag{3.2.61}$$

is exact. Furthermore, by Lemma 3.2.6, we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (V_0)_{T,\Lambda} & \longrightarrow & (V_1)_{T,\Lambda} & \longrightarrow & (V_1/V_0)_{T,\Lambda} & \longrightarrow & 0 \\ & & {\scriptstyle\alpha}\big\downarrow & & {\scriptstyle\beta}\big\downarrow & & {\scriptstyle\gamma}\big\downarrow & & \\ 0 & \longrightarrow & (V_0)_{T,\Lambda} & \longrightarrow & V_{T,\Lambda} & \longrightarrow & (V/V_0)_{T,\Lambda} & \longrightarrow & 0 \end{array} \tag{3.2.62}$$

The exact sequence $\ker(\alpha) \to \ker(\beta) \to \ker(\gamma)$ reads $0 \to \ker(\beta) \to 0$ by the exactness of (3.2.61). It follows that the map $(V_1)_{T,\Lambda} \to V_{T,\Lambda}$ in (3.2.59) is injective. The other parts of the sequence are exact by (3.2.30). This concludes the proof. $\square$

3.2.3 Split Waldspurger modules of reducible principal series

Reducible principal series of $\mathrm{GL}(2, F)$ do not belong to the category $\mathcal{C}$. These cases require some special arguments. Let us choose $\beta = \begin{bmatrix} & 1/2 \\ 1/2 & \end{bmatrix}$. It follows that $\Delta = (-1/2, 1/2)$ and $L = F \oplus F$.

Lemma 3.2.8.

$$\dim(\sigma\nu^{-1/2} \times \sigma\nu^{1/2})_{T,\Lambda} = \begin{cases} 2 & \textit{if } \Lambda = \sigma \circ \mathrm{N}_{L/F}, \\ 1 & \textit{otherwise.} \end{cases} \tag{3.2.63}$$

Proof. Let us consider the short sequence

$$0 \longrightarrow \sigma 1_{F^\times} \longrightarrow \sigma\nu^{-1/2} \times \sigma\nu^{1/2} \longrightarrow \sigma \mathrm{St}_{\mathrm{GL}(2)} \longrightarrow 0$$

But $0 \subset \sigma 1_{F^\times} \subset \sigma\nu^{-1/2} \times \sigma\nu^{1/2}$ satisfies the condition (3.2.34). Let $U \subset \sigma 1_{F^\times}$ be as in Lemma 3.2.4. By Lemma 3.2.4,

$$U \cap (\sigma 1_{F^\times})(T, \Lambda) = U \cap (\sigma\nu^{-1/2} \times \sigma\nu^{1/2})(T, \Lambda).$$

If $\sigma = \Lambda \circ \mathrm{N}_{L/F}$, then $U = \sigma 1_{F^\times}$ and $(\sigma 1_{F^\times})(T, \Lambda) = 0$. Otherwise, if $\sigma \neq \Lambda \circ \mathrm{N}_{L/F}$, then $(\sigma 1_{F^\times})(T, \Lambda) = \sigma 1_{F^\times}$ is one dimensional. In either case, we have

$$0 \longrightarrow (\sigma 1_{F^\times})_{T,\Lambda} \longrightarrow (\sigma\nu^{-1/2} \times \sigma\nu^{1/2})_{T,\Lambda},$$

so

$$0 \longrightarrow (\sigma 1_{F^\times})_{T,\Lambda} \longrightarrow (\sigma\nu^{-1/2} \times \sigma\nu^{1/2})_{T,\Lambda} \longrightarrow (\sigma\mathrm{St}_{\mathrm{GL}(2)})_{T,\Lambda} \longrightarrow 0.$$

Hence, $\dim(\sigma\nu^{-1/2} \times \sigma\nu^{1/2})_{T,\Lambda} = \dim(\sigma 1_{F^\times})_{T,\Lambda} + \dim(\sigma\mathrm{St}_{\mathrm{GL}(2)})_{T,\Lambda}$. But

$$\dim(\sigma 1_{F^\times})_{T,\Lambda} = \begin{cases} 1 & \text{if } \Lambda = \sigma \circ \mathrm{N}_{L/F}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.2.64}$$

This proves the lemma. □

Lemma 3.2.9. $\dim(\sigma\nu^{1/2} \times \sigma\nu^{-1/2})_{T,\Lambda} = 1$, *for every character* Λ.

Proof. We choose a character ψ of F, which has conductor $\mathfrak{o}$. Let us consider the short exact sequence

$$0 \longrightarrow \sigma\mathrm{St}_{\mathrm{GL}(2)} \longrightarrow \sigma\nu^{1/2} \times \sigma\nu^{-1/2} \longrightarrow \sigma 1_{F^\times} \longrightarrow 0.$$

We have two cases:

Case 1: $\Lambda \neq \sigma \circ \mathrm{N}_{L/F}$. Similarly, $0 \subset \sigma\mathrm{St}_{\mathrm{GL}(2)} \subset \sigma\nu^{1/2} \times \sigma\nu^{-1/2}$ satisfies the condition (3.2.34). By Lemma 3.2.6, we have the short exact sequence

$$0 \longrightarrow (\sigma\mathrm{St}_{\mathrm{GL}(2)})_{T,\Lambda} \longrightarrow (\sigma\nu^{1/2} \times \sigma\nu^{-1/2})_{T,\Lambda} \longrightarrow (\sigma 1_{F^\times})_{T,\Lambda} \longrightarrow 0.$$

But $(1_{F^\times})_{T,\Lambda} = 0$, so $(\sigma\nu^{1/2} \times \sigma\nu^{-1/2})_{T,\Lambda} \cong (\sigma\mathrm{St}_{\mathrm{GL}(2)})_{T,\Lambda}$ is one dimensional. Since $\sigma\mathrm{St}_{\mathrm{GL}(2)}$ is infinitely dimensional, irreducible, admissible representation of $\mathrm{GL}(2,F)$, $U \subset \sigma\mathrm{St}_{\mathrm{GL}(2)}$ is not zero by Lemma 3.2.5.

Case 2: $\Lambda = \sigma \circ \mathrm{N}_{L/F}$. Our first step is to calculate $(\nu^{1/2} \times \nu^{-1/2})_{T,1_{F^\times}\circ\mathrm{N}}$. By

Proposition 2.1.2 of [13], $\nu^{1/2} \times \nu^{-1/2}$ has a spherical vector

$$f_0(g) = |ad^{-1}|, \text{ if } g \in \left[\begin{smallmatrix} a & * \\ & d \end{smallmatrix}\right]\mathrm{GL}(2, \mathfrak{o}). \tag{3.2.65}$$

Since $\mathrm{St}_{\mathrm{GL}(2)}$ has no spherical vectors, $f_0 \notin \mathrm{St}_{\mathrm{GL}(2)}$.

Let us define a Whittaker functional $l : \mathrm{St}_{\mathrm{GL}(2)} \to \mathbb{C}$ by

$$f \mapsto \lim_{N \to +\infty} \int_{\mathfrak{p}^{-N}} f(\left[\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & a \\ & 1 \end{smallmatrix}\right])\psi(-a)da, \tag{3.2.66}$$

for any $f \in \mathrm{St}_{\mathrm{GL}(2)}$. We define $W_f : \mathrm{GL}(2, F) \to \mathbb{C}$ such that $g \mapsto l(\pi(g)f)$, $f \in \mathrm{St}_{\mathrm{GL}(2)}$. Let us define an explicit Waldspurger functional of $\mathrm{St}_{\mathrm{GL}(2)}$ as follows

$$L(f) := \int_{F^\times} W_f(\left[\begin{smallmatrix} x & \\ & 1 \end{smallmatrix}\right])d^\times x, \tag{3.2.67}$$

for any $f \in \mathrm{St}_{\mathrm{GL}(2)}$. It is known that for any $f \in \mathrm{St}_{\mathrm{GL}(2)}$, $W_f(\left[\begin{smallmatrix} x & \\ & 1 \end{smallmatrix}\right])$ has a bounded support and $W_f(\left[\begin{smallmatrix} x & \\ & 1 \end{smallmatrix}\right]) = C|x|$, for $|x| < \epsilon(f)$ and some constant $C = C(f) \in \mathbb{C}$. It follows $L(f) < +\infty$, i.e, L is well-defined.

We claim that $L(f_1) \neq 0$, where $f_1 := \pi(\left[\begin{smallmatrix} \varpi & \\ & 1 \end{smallmatrix}\right])f_0 - f_0 \in \mathrm{St}_{\mathrm{GL}(2)}$.

$$L(f_1) = \int_{F^\times} W_{f_1}(\left[\begin{smallmatrix} x & \\ & 1 \end{smallmatrix}\right])d^\times x = \sum_{m \in \mathbb{Z}} \int_{\varpi^m \mathfrak{o}_{F^\times}} W_{f_1}(\left[\begin{smallmatrix} x & \\ & 1 \end{smallmatrix}\right])d^\times x = \sum_{m \in \mathbb{Z}} I_m,$$

where

$$\begin{aligned} I_m &= \int_{\varpi^m \mathfrak{o}_{F^\times}} W_{f_1}(\left[\begin{smallmatrix} x & \\ & 1 \end{smallmatrix}\right])d^\times x \\ &= \int_{\varpi^m \mathfrak{o}_{F^\times}} \big(\lim_{N \to +\infty} \int_{\mathfrak{p}^{-N}} f_1(\left[\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & a \\ & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} x & \\ & 1 \end{smallmatrix}\right])\psi(-a)da\big)d^\times x. \end{aligned}$$

Let us consider $N \gg 0$ such that $-N < v(x) = m$, then

$$\begin{aligned}
&\int_{\mathfrak{p}^{-N}} f_1(\left[\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & a \\ & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} x & \\ & 1 \end{smallmatrix}\right])\psi(-a)da = \int_{\mathfrak{p}^{-N}} f_1(\left[\begin{smallmatrix} & -1 \\ x & a \end{smallmatrix}\right])\psi(-a)da \\
&= \int_{\mathfrak{p}^{-N}} (\pi(\left[\begin{smallmatrix} \varpi & \\ & 1 \end{smallmatrix}\right])f_0 - f_0)(\left[\begin{smallmatrix} & -1 \\ x & a \end{smallmatrix}\right])\psi(-a)da \\
&= \sum_{n=-N}^{+\infty} \int_{\varpi^n \mathfrak{o}^\times} \big(f_0(\left[\begin{smallmatrix} & -1 \\ x\varpi & a \end{smallmatrix}\right]) - f_0(\left[\begin{smallmatrix} & -1 \\ x & a \end{smallmatrix}\right])\big)\psi(-a)da \\
&= \sum_{n=-N}^{+\infty} \int_{\varpi^n \mathfrak{o}^\times} \big(f_0(\left[\begin{smallmatrix} x\varpi a^{-1} & -1 \\ & a \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & \\ x\varpi a^{-1} & 1 \end{smallmatrix}\right]) - f_0(\left[\begin{smallmatrix} xa^{-1} & -1 \\ & a \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & \\ xa^{-1} & 1 \end{smallmatrix}\right])\big)\psi(-a)da.
\end{aligned}$$

Since

$$f_0(\left[\begin{smallmatrix} 1 & \\ y & 1 \end{smallmatrix}\right]) = \begin{cases} |y|^{-2} & \text{if } v(y) \leq 0, \\ 1 & \text{if } v(y) > 0; \end{cases} \tag{3.2.68}$$

then

$$\begin{aligned}
&\int_{\mathfrak{p}^{-N}} f_1(\left[\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & a \\ & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} x & \\ & 1 \end{smallmatrix}\right])\psi(-a)da \\
&= \sum_{n=-N}^{+\infty} \int_{\varpi^n \mathfrak{o}^\times} \big(q^{2n-m-1} f_0(\left[\begin{smallmatrix} 1 & \\ x\varpi a^{-1} & 1 \end{smallmatrix}\right]) - q^{2n-m} f_0(\left[\begin{smallmatrix} 1 & \\ xa^{-1} & 1 \end{smallmatrix}\right])\big)\psi(-a)da \\
&= \sum_{n=-N}^{m} \int_{\varpi^n \mathfrak{o}^\times} \big(q^{2n-m-1} f_0(\left[\begin{smallmatrix} 1 & \\ x\varpi a^{-1} & 1 \end{smallmatrix}\right]) - q^{2n-m} f_0(\left[\begin{smallmatrix} 1 & \\ xa^{-1} & 1 \end{smallmatrix}\right])\big)\psi(-a)da+ \\
&+ \int_{\varpi^{m+1} \mathfrak{o}^\times} \big(q^{m+1} f_0(\left[\begin{smallmatrix} 1 & \\ x\varpi a^{-1} & 1 \end{smallmatrix}\right]) - q^{m+2} f_0(\left[\begin{smallmatrix} 1 & \\ xa^{-1} & 1 \end{smallmatrix}\right])\big)\psi(-a)da+ \\
&+ \sum_{n=m+2}^{+\infty} \int_{\varpi^n \mathfrak{o}^\times} \big(q^{2n-m-1} f_0(\left[\begin{smallmatrix} 1 & \\ x\varpi a^{-1} & 1 \end{smallmatrix}\right]) - q^{2n-m} f_0(\left[\begin{smallmatrix} 1 & \\ xa^{-1} & 1 \end{smallmatrix}\right])\big)\psi(-a)da \\
&= \sum_{n=-N}^{m} (q^{2n-m-1} - q^{2n-m}) \int_{\varpi^n \mathfrak{o}^\times} \psi(-a)da + (q^{m+1} - q^m) \int_{\varpi^{m+1} \mathfrak{o}^\times} \psi(-a)da \\
&+ \sum_{n=m+2}^{+\infty} (q^{m+1} - q^m) \int_{\varpi^n \mathfrak{o}^\times} \psi(-a)da
\end{aligned}$$

$$= \sum_{n=-N}^{m} (q^{2n-m-1} - q^{2n-m}) \int_{\varpi^n \mathfrak{o}^\times} \psi(-a) da + \sum_{n=m+1}^{+\infty} (q^{m+1} - q^m) \int_{\varpi^n \mathfrak{o}^\times} \psi(-a) da.$$

On the other hand, one can show

$$\int_{\varpi^n \mathfrak{o}^\times} \psi(-a) da = \begin{cases} \frac{1}{q^n} - \frac{1}{q^{n+1}} & \text{if } n \geq 0, \\ -1 & \text{if } n = -1, \\ 0 & \text{if } n < -1. \end{cases} \tag{3.2.69}$$

Now we consider the following cases:

- $\underline{m < -1}$. We have

$$\sum_{n=-N}^{m} (q^{2n-m-1} - q^{2n-m}) \int_{\varpi^n \mathfrak{o}^\times} \psi(-a) da = 0,$$

  and

$$\begin{aligned} \sum_{n=m+1}^{+\infty} (q^{m+1} - q^m) \int_{\varpi^n \mathfrak{o}^\times} \psi(-a) da &= (q^{m+1} - q^m)\Big(\int_{\varpi^{-1} \mathfrak{o}^\times} \psi(-a) da + \\ &\quad + \sum_{n=0}^{+\infty} \int_{\varpi^n \mathfrak{o}^\times} \psi(-a) da\Big) \\ &= (q^{m+1} - q^m)\Big(-1 + \sum_{n=0}^{+\infty} (\frac{1}{q^n} - \frac{1}{q^{n+1}})\Big) \\ &= 0. \end{aligned}$$

  Hence, $I_m = 0$ if $m < -1$.

- $\underline{m=-1}$. In this case,

$$\sum_{n=-N}^{-1}(q^{-2}-q^{-1})\int_{\varpi^{-1}\mathfrak{o}^\times}\psi(-a)da = q^{-2}-q^{-1},$$

and

$$\sum_{n=0}^{+\infty}(1-q^{-1})\int_{\varpi^n\mathfrak{o}^\times}\psi(-a)da = (1-q^{-1})\sum_{n=0}^{+\infty}(\frac{1}{q^n}-\frac{1}{q^{n+1}}) = 1-q^{-1}.$$

Hence, $I_m = (1-q^{-1})(1+q^{-2})$ if $m=-1$.

- $\underline{m \geq 0}$. We have

$$\begin{aligned}
&\sum_{n=-N}^{m}(q^{2n-m-1}-q^{2n-m})\int_{\varpi^n\mathfrak{o}^\times}\psi(-a)da = \\
&= \sum_{n=-N}^{-2}(q^{2n-m-1}-q^{2n-m})\int_{\varpi^n\mathfrak{o}^\times}\psi(-a)da + (q^{-m-3}-q^{-m-2})\times \\
&\times \int_{\varpi^{-1}\mathfrak{o}^\times}\psi(-a)da + \sum_{n=0}^{m}(q^{2n-m-1}-q^{2n-m})\int_{\varpi^n\mathfrak{o}^\times}\psi(-a)da \\
&= 0 + (-q^{-m-3}+q^{-m-2}) + \sum_{n=0}^{m}(q^{2n-m-1}-q^{2n-m})(\frac{1}{q^n}-\frac{1}{q^{n+1}}) \\
&= -1+q^{-1}+q^{-m-1}-q^{-m-3},
\end{aligned}$$

and

$$\begin{aligned}
\sum_{n=m+1}^{+\infty}(q^{m+1}-q^m)\int_{\varpi^n\mathfrak{o}^\times}\psi(-a)da &= \sum_{n=m+1}^{+\infty}(q^{m+1}-q^m)(\frac{1}{q^n}-\frac{1}{q^{n+1}}) \\
&= (q^{m+1}-q^m)q^{-m-1} = 1-q^{-1}.
\end{aligned}$$

Hence, $I_m = (q^{-m})[(-1+q^{-1}+q^{-m-1}-q^{-m-3})+(1-q^{-1})] = q^{-2m-1}-q^{-2m-3}$

if $m \geq 0$.

It follows

$$L(f_1) = \sum_{m \in \mathbb{Z}} I_m = (1 - q^{-1})(1 + q^{-2}) + \sum_{m=0}^{+\infty} (q^{-m-1} - q^{-m-3}) = 1 + \frac{1}{q^2} \neq 0.$$

By the uniqueness of Waldspurger models, $f_1 \notin \mathrm{St}_{\mathrm{GL}(2)}(T, 1_{F^\times} \circ \mathrm{N})$. Hence,

$$(\mathrm{St}_{\mathrm{GL}(2)})_{T, 1_{F^\times} \circ \mathrm{N}} = \mathbb{C} \cdot \bar{f}_1. \tag{3.2.70}$$

But $f_1 = \pi([\begin{smallmatrix} \varpi & \\ & 1 \end{smallmatrix}])f_0 - f_0$, then $f_1 \in (\nu^{1/2} \times \nu^{-1/2})(T, 1_{F^\times} \circ \mathrm{N})$. Hence, the embedding $\mathrm{St}_{\mathrm{GL}(2)} \to \nu^{1/2} \times \nu^{-1/2}$ induces the zero map $(\mathrm{St}_{\mathrm{GL}(2)})_{T, 1_{F^\times} \circ \mathrm{N}} \to \{0\} \subset (\nu^{1/2} \times \nu^{-1/2})_{T, 1_{F^\times} \circ \mathrm{N}}$. Now the exact sequence

$$(\mathrm{St}_{\mathrm{GL}(2)})_{T, 1_{F^\times} \circ \mathrm{N}} \longrightarrow (\nu^{1/2} \times \nu^{-1/2})_{T, 1_{F^\times} \circ \mathrm{N}} \longrightarrow (1_{F^\times})_{T, 1_{F^\times} \circ \mathrm{N}} \longrightarrow 0$$

implies that $(\nu^{1/2} \times \nu^{-1/2})_{T, 1_{F^\times} \circ \mathrm{N}}$ is a one dimensional vector space.

The next step is to show that $(\sigma\nu^{1/2} \times \sigma\nu^{-1/2})_{T, \sigma \circ \mathrm{N}}$ is also one dimensional. In order to see that, we need the following fact:

- Let (π, V) be a representation of $GL(2, F)$, Λ be a character of T and σ be a character of $F^\times$. Then $(\sigma V)_{T, \Lambda_\sigma} \cong V_{T, \Lambda}$ as vector spaces, where $(\sigma\pi, \sigma V)$ is a twisted representation of (π, V) and Λ_σ is the twisted character of Λ by σ.

The isomorphism is induced by the identical map $V \to \sigma V$ such that $v \mapsto v$. In fact, $\pi(t)v - \Lambda(t)v = (\sigma\pi)(t)w - \Lambda_\sigma(t)w$, where $w = (\sigma \circ \det)^{-1}(t)v$. It follows

$$V(T, \Lambda) = (\sigma V)(T, \Lambda_\sigma).$$

Applying to our case with $V = \nu^{1/2} \times \nu^{-1/2}$ and $\Lambda = 1_{F^\times} \circ \mathrm{N}$, we see that $(\sigma\nu^{1/2} \times \sigma\nu^{-1/2})_{T,\sigma\circ\mathrm{N}}$ is one dimensional. $\square$

3.3 Jacquet-Waldspurger modules

Recall the groups N, T defined in (2.1.2) resp. (2.1.9), and the algebra $A_\beta \subset M_2(F)$ defined in (2.1.6). Let (π, V) be an admissible representation of $\mathrm{GSp}(4, F)$. We now consider

$$V(N, T, \Lambda) = \langle \pi(tn)v - \Lambda(t)v \,:\, v \in V,\, t \in T,\, n \in N\rangle \text{ and } V_{N,T,\Lambda} = V/V(N, T, \Lambda). \tag{3.3.1}$$

Evidently, there is a surjective map $V_N \to V_{N,T,\Lambda}$ which induces an isomorphism

$$(V_N)_{T,\Lambda} \cong V_{N,T,\Lambda}. \tag{3.3.2}$$

Here, on the left we use the notation (3.2.1) for the $\mathrm{GL}(2, F)$-module V_N. Note that, in view of (2.1.8), we have to embed $\mathrm{GL}(2, F)$ into $\mathrm{GSp}(4, F)$ via the map

$$\mathrm{GL}(2, F) \ni g \longmapsto \begin{bmatrix} g & \\ & \det(g)\,{}^tg^{-1} \end{bmatrix}, \tag{3.3.3}$$

and consider V_N a $\mathrm{GL}(2, F)$-module via this embedding. We call $V_{N,T,\Lambda}$ the *Jacquet-Waldspurger module* of π. This module retains an action of $F^\times$, coming from the action of the group $\{\mathrm{diag}(x,x,1,1) : x \in F^\times\}$ on V. The map $V \mapsto V_{N,T,\Lambda}$ defines a functor, called *Jacquet-Waldspurger functor*, from the category of admissible $\mathrm{GSp}(4,F)$-representations to the category of $F^\times$-modules.

Lemma 3.3.1. *Let V, V', V'' be admissible representations of $\mathrm{GSp}(4,F)$.*

i) If $V = V' \oplus V''$ is a direct sum, then

$$V_{N,T,\Lambda} = V'_{N,T,\Lambda} \oplus V''_{N,T,\Lambda}. \tag{3.3.4}$$

ii) The Jacquet-Waldspurger functor is right exact, i.e, if $0 \to V' \to V \to V'' \to 0$ is exact, then

$$V'_{N,T,\Lambda} \longrightarrow V_{N,T,\Lambda} \longrightarrow V''_{N,T,\Lambda} \longrightarrow 0 \tag{3.3.5}$$

is exact. Moreover, if we are in the non-split case, then the Jacquet - Waldspurger functor is exact.

Proof. These are general properties of Jacquet-type functors. See Proposition 2.35 of [2]. □

Lemma 3.3.2. *Let (π, V) be an admissible representation of $\mathrm{GSp}(4,F)$ of finite length. Then the $F^\times$-module $V_{N,T,\Lambda}$ is finite-dimensional. More precisely, if n is the length of the $\mathrm{GL}(2,F)$-module V_N, then $\dim V_{N,T,\Lambda} \le n$.*

Proof. The proof is by induction on n. If $n = 1$, then V_N is an irreducible, admissible representation of $\mathrm{GL}(2,F)$. In this case the assertion follows from (3.2.3).

Assume that $n > 1$. Let V' be a submodule of V_N of length $n-1$. Then $V'' := V_N/V'$ is irreducible. By (3.3.5), we have an exact sequence

$$V'_{T,\Lambda} \xrightarrow{\alpha} V_{N,T,\Lambda} \longrightarrow V''_{T,\Lambda} \longrightarrow 0. \tag{3.3.6}$$

By induction and (3.2.3), it follows that

$$\dim V_{N,T,\Lambda} = \dim \operatorname{im}(\alpha) + \dim V''_{T,\Lambda} \leq n - 1 + 1 = n. \tag{3.3.7}$$

This concludes the proof. □

Theorem 3.3.3. *Let (π, V) be an admissible, irreducible representation of* $\mathrm{GSp}(4, F)$. *Assume that we are in the non-split case. Then the semisimplications of $V_{N,T,\Lambda}$ are as given in Table 3.3.*

Proof. Since we are in the non-split case, the quadratic extension L is a field. Then the semisimplifications of the $V_{N,T,\Lambda}$ can easily be calculated from V_N using (3.3.2). By Lemma 3.3.1 (ii), in the non-split case the Waldspurger functor is exact. Therefore, to calculate the $V_{N,T,\Lambda}$, we can simply take $(\tau \otimes \sigma)_{T,\Lambda}$ for each constituent $\tau \otimes \sigma$ occurring in Table 3.1. If $\tau_{T,\Lambda}$ is one-dimensional, then $(\tau \otimes \sigma)_{T,\Lambda} = \sigma 1_{F^\times}$ as an $F^\times$-module, and if $\tau_{T,\Lambda} = 0$, then $(\tau \otimes \sigma)_{T,\Lambda} = 0$. We have listed the semisimplifications of the $V_{N,T,\Lambda}$ for all irreducible, admissible representations in Table 3.3. □

We denote

$$S_1 = \{\begin{bmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & \\ & & -x & 1 \end{bmatrix} : x \in F\}, \qquad S = \{\begin{bmatrix} 1 & x & * & * \\ & 1 & * & y \\ & & 1 & \\ & & -x & 1 \end{bmatrix} : x, y \in F\}.$$

Hence, $S = S_1 N$ is the unipotent radical of the Borel parabolic subgroup. We fix a non-trivial character ψ_0 of F. Let us define a character ψ_{c_1,c_2} of S by

$$\psi_{c_1,c_2}(\begin{bmatrix} 1 & x & * & * \\ & 1 & * & y \\ & & 1 & \\ & & -x & 1 \end{bmatrix}) = \psi_0(c_1 x + c_2 y), \tag{3.3.8}$$

where $c_1,\ c_2 \in F$. We let $\psi_1 = \psi_{1,0}$. Similarly, we define the twisted Jacquet module of (π, V) associated to ψ_1 as follows

$$V(S, \psi_1) = \langle \pi(s)v - \psi_1(s)v \ :\ v \in V,\ s \in S \rangle \qquad \text{and} \qquad V_{S,\psi_1} = V/V(S, \psi_1).$$

In fact, V_{S,ψ_1} admits an action of H.

Lemma 3.3.4. *Let (π, V) be a generic representation of* $\mathrm{GSp}(4, F)$*, S and ψ_1 be as above. Then the algebraic structure of V_{S,ψ_1} can be obtained in the Table 3.2.*

$$V_{S,\psi_1} = \begin{cases} \nu^{3/2}\chi_1\chi_2\sigma \oplus \nu^{3/2}\chi_1\sigma \oplus \nu^{3/2}\chi_2\sigma \oplus \nu^{3/2}\sigma & \text{if } \chi_1\chi_2, \chi_1, \chi_2, 1 \text{ are} \\ & \text{pairwise different,} \\ \nu^{3/2}\chi^2\sigma \oplus (\nu^{3/2}\chi\sigma)[2] \oplus \nu^{3/2}\sigma & \text{if } \chi := \chi_1 = \chi_2 \neq 1,\ \chi^2 \neq 1, \\ (\nu^{3/2}\chi\sigma)[2] \oplus (\nu^{3/2}\sigma)[2] & \text{if } \chi := \chi_1 = \chi_2 \neq 1,\ \chi^2 = 1, \\ (\nu^{3/2}\chi\sigma)[2] \oplus (\nu^{3/2}\sigma)[2] & \text{if } \{\chi_1, \chi_2\} = \{\chi \neq 1, 1\} \\ (\nu^{3/2}\sigma)[4] & \text{if } \chi_1 = \chi_2 = 1. \end{cases} \tag{3.3.9}$$

Proof. It is a consequence of Theorem 5.4 and Table 3 in [15]. □

We have the following easy-to-prove lemma to determine the algebraic structure of the split Jacquet-Waldspurger modules in Theorem 3.3.6.

Table 3.2: The algebraic structure of V_{S,ψ_1}

		representation		V_{S,ψ_1}
I		$\chi_1 \times \chi_2 \rtimes \sigma$		see (3.3.9)
II	a	$\chi\mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$	$\chi^2 \neq 1$	$\nu^2\chi\sigma \oplus \nu^{3/2}\chi^2\sigma \oplus \nu^{3/2}\sigma$
			$\chi^2 = 1$	$\nu^2\chi\sigma \oplus (\nu^{3/2}\sigma)[2]$
III	a	$\chi \rtimes \sigma\mathrm{St}_{\mathrm{GSp}(2)}$		$\chi\nu^2\sigma \oplus \nu^2\sigma$
IV	a	$\sigma\mathrm{St}_{\mathrm{GSp}(4)}$		$\nu^3\sigma$
V	a	$\delta([\xi, \nu\xi], \nu^{-1/2}\sigma)$		$\nu^2\sigma \oplus \xi\nu^2\sigma$
VI	a	$\tau(S, \nu^{-1/2}\sigma)$		$(\nu^2\sigma)[2]$
VII		$\chi \rtimes \pi$		0
VIII	a	$\tau(S, \pi)$		0
IX	a	$\delta(\nu\xi, \nu^{-1/2}\pi(\mu))$		0
X		$\pi \rtimes \sigma$	$\omega_\pi \neq 1$	$\nu^{3/2}\omega_\pi\sigma \oplus \nu^{3/2}\sigma$
			$\omega_\pi = 1$	$(\nu^{3/2}\sigma)[2]$
XI	a	$\delta(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$		$\nu^2\sigma$
		generic supercuspidal		0

Lemma 3.3.5. *Let $\tau = \sum_{i=1}^{n} \tau_i \otimes \sigma_i$ be a decomposition of $\mathrm{GL}(2,F) \times H$-modules, where τ_i are irreducible, admissible representations of $\mathrm{GL}(2,F)$, σ_i, and σ be characters of H. Then $\tau_{H,\sigma} = \sum_{j=1}^{k} \tau_{i_j}$ as $\mathrm{GL}(2,F)$-modules, for some $k \leq n$ and $\{i_1, i_2, ..., i_k\} \subset \{1, 2, ..., n\}$.*

Theorem 3.3.6. *We have Table 3.4 of the algebraic decompositions of split Jacquet-Waldspurger modules for all non-supercuspidal, irreducible, admissible representations of $\mathrm{GSp}(4,F)$. For type I, we have to distinguish various cases, depending on the regularity of the inducing character:*

$$V_{N,T,\Lambda} = \begin{cases} \nu^{3/2}\chi_1\chi_2\sigma \oplus \nu^{3/2}\chi_1\sigma \oplus \nu^{3/2}\chi_2\sigma \oplus \nu^{3/2}\sigma & \textit{if } \chi_1\chi_2, \chi_1, \chi_2, 1 \textit{ are} \\ & \textit{pairwise different,} \\ \nu^{3/2}\chi^2\sigma \oplus (\nu^{3/2}\chi\sigma)[2] \oplus \nu^{3/2}\sigma & \textit{if } \chi := \chi_1 = \chi_2 \neq 1,\ \chi^2 \neq 1, \\ (\nu^{3/2}\chi\sigma)[2] \oplus (\nu^{3/2}\sigma)[2] & \textit{if } \chi := \chi_1 = \chi_2 \neq 1,\ \chi^2 = 1, \\ (\nu^{3/2}\chi\sigma)[2] \oplus (\nu^{3/2}\sigma)[2] & \textit{if } \{\chi_1, \chi_2\} = \{\chi \neq 1, 1\} \\ (\nu^{3/2}\sigma)[4] & \textit{if } \chi_1 = \chi_2 = 1. \end{cases} \tag{3.3.10}$$

Proof. Recall that we choose $\beta = \begin{bmatrix} & 1/2 \\ 1/2 & \end{bmatrix}$ in the split case. We will treat two cases of the theorem, namely non-generic and generic representations, in different ways. Notice that the Bessel models do not exist for types IVd, Vd, VIb and IXb.

i) Non-generic cases: In this case, we consider the Jacquet module of (π, V) of types IIb, IIIb, IVb, IVc, Vb, Vc, VIc, VId, VIIIb and XIb as given in

Table 3.3: The semisimplifications of Jacquet-Waldspurger modules. It is assumed that L is a field, and that the representation of $\mathrm{GSp}(4,F)$ admits a (Λ,β)-Bessel functional. An entry "—" indicates that no such Bessel functional exists.

		representation	semisimplification of $V_{N,T,\Lambda}$
I		$\chi_1\times\chi_2\rtimes\sigma$ (irreducible)	$\nu^{3/2}\chi_1\chi_2\sigma 1_{F^\times}+\nu^{3/2}\sigma 1_{F^\times}$ $+\nu^{3/2}\chi_1\sigma 1_{F^\times}+\nu^{3/2}\chi_2\sigma 1_{F^\times}$
II	a	$\chi\mathrm{St}_{\mathrm{GL}(2)}\rtimes\sigma$	$\nu^{3/2}\chi^2\sigma 1_{F^\times}+\nu^{3/2}\sigma 1_{F^\times}+\nu^2\chi\sigma 1_{F^\times}$
	b	$\chi 1_{\mathrm{GL}(2)}\rtimes\sigma$	$\nu^{3/2}\chi^2\sigma 1_{F^\times}+\nu^{3/2}\sigma 1_{F^\times}+\nu\chi\sigma 1_{F^\times}$
III	a	$\chi\rtimes\sigma\mathrm{St}_{\mathrm{GSp}(2)}$	$\chi\nu^2\sigma 1_{F^\times}+\nu^2\sigma 1_{F^\times}$
	b	$\chi\rtimes\sigma 1_{\mathrm{GSp}(2)}$	—
IV	a	$\sigma\mathrm{St}_{\mathrm{GSp}(4)}$	$\nu^3\sigma 1_{F^\times}$
	b	$L(\nu^2,\nu^{-1}\sigma\mathrm{St}_{\mathrm{GSp}(2)})$	$\nu^3\sigma 1_{F^\times}+\nu\sigma 1_{F^\times}$
	c	$L(\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)},\nu^{-3/2}\sigma)$	—
	d	$\sigma 1_{\mathrm{GSp}(4)}$	—
V	a	$\delta([\xi,\nu\xi],\nu^{-1/2}\sigma)$	$\nu^2\sigma 1_{F^\times}+\xi\nu^2\sigma 1_{F^\times}$
	b	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma)$	$\nu\sigma 1_{F^\times}+\xi\nu^2\sigma 1_{F^\times}$
	c	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\xi\sigma)$	$\xi\nu\sigma 1_{F^\times}+\nu^2\sigma 1_{F^\times}$
	d	$L(\nu\xi,\xi\rtimes\nu^{-1/2}\sigma)$	$\xi\nu\sigma 1_{F^\times}+\nu\sigma 1_{F^\times}$
VI	a	$\tau(S,\nu^{-1/2}\sigma)$	$2\cdot(\nu^2\sigma 1_{F^\times})$
	b	$\tau(T,\nu^{-1/2}\sigma)$	$\nu^2\sigma 1_{F^\times}$
	c	$L(\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma)$	—
	d	$L(\nu,1_{F^\times}\rtimes\nu^{-1/2}\sigma)$	—
VII		$\chi\rtimes\pi$	0

		representation	semisimplification of $V_{N,T,\Lambda}$
VIII	a	$\tau(S,\pi)$	0
	b	$\tau(T,\pi)$	0
IX	a	$\delta(\nu\xi,\nu^{-1/2}\pi(\mu))$	0
	b	$L(\nu\xi,\nu^{-1/2}\pi(\mu))$	0
X		$\pi \rtimes \sigma$	$\nu^{3/2}\omega_\pi\sigma 1_{F^\times} + \nu^{3/2}\sigma 1_{F^\times}$
XI	a	$\delta(\nu^{1/2}\pi,\nu^{-1/2}\sigma)$	$\nu^2\sigma 1_{F^\times}$
	b	$L(\nu^{1/2}\pi,\nu^{-1/2}\sigma)$	$\nu\sigma 1_{F^\times}$
		supercuspidal	0

Table 3.1, is $\sum \tau_i \otimes \chi_i$ with the χ_i pairwise different. Then

$$V_N = \bigoplus_{i=1}^{n} V_N(\chi_i), \tag{3.3.11}$$

where

$$V_N(\chi_i) = \{v \in V_N \mid a.v = \chi_i(a)v \text{ for all } a \in F^\times\}.$$

Since the action of $F^\times$ commutes with the action of $\mathrm{GL}(2,F)$, the $V_N(\chi_i)$ are $\mathrm{GL}(2,F)$-invariant. Hence, (3.3.11) is a direct sum of $\mathrm{GL}(2,F) \times F^\times$-modules (not only of $F^\times$-modules). Since the only $\mathrm{GL}(2,F)\times F^\times$-subquotients of V_N are the $\tau_i \otimes \chi_i$, it follows that $V_N(\chi_i) \cong \tau_i \otimes \chi_i$. Therefore $V_N = \bigoplus \tau_i \otimes \chi_i$ is a direct sum. We can then use Lemma 3.3.1 ii) and obtain

$$V_{N,T,\Lambda} = \oplus((\tau_i)_{T,\Lambda} \otimes \chi_i). \tag{3.3.12}$$

Of course, $\dim(\tau_i)_{T,\Lambda} = 1$ if τ_i is infinite-dimensional, and $\dim(\tau_i)_{T,\Lambda} \in \{0,1\}$ if τ_i is one-dimensional.

Table 3.4: The algebraic structure of Jacquet-Waldspurger modules in the split case. It is assumed that $L = F \times F$, and that the representation of $\mathrm{GSp}(4, F)$ admits a (Λ, β)-Bessel functional. An entry "—" indicates that no such Bessel functional exists.

		representation		$V_{N,T,\Lambda}$
I		$\chi_1 \times \chi_2 \rtimes \sigma$		see (3.3.10)
II	a	$\chi\mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$	$\chi^2 \neq 1$	$\nu^2\chi\sigma \oplus \nu^{3/2}\chi^2\sigma \oplus \nu^{3/2}\sigma$
			$\chi^2 = 1$	$\nu^2\chi\sigma \oplus (\nu^{3/2}\sigma)[2]$
	b	$\chi 1_{\mathrm{GL}(2)} \rtimes \sigma$	$\chi^2 \neq 1$	$\nu\chi\sigma \oplus \nu^{3/2}\chi^2\sigma \oplus \nu^{3/2}\sigma$
			$\chi^2 = 1$	$\nu\chi\sigma \oplus (\nu^{3/2}\sigma)[2]$
III	a	$\chi \rtimes \sigma\mathrm{St}_{\mathrm{GSp}(2)}$		$\chi\nu^2\sigma \oplus \nu^2\sigma$
	b	$\chi \rtimes \sigma 1_{\mathrm{GSp}(2)}$		$\chi\nu\sigma \oplus \nu\sigma$
IV	a	$\sigma\mathrm{St}_{\mathrm{GSp}(4)}$		$\nu^3\sigma$
	b	$L(\nu^2, \nu^{-1}\sigma\mathrm{St}_{\mathrm{GSp}(2)})$		$\nu^3\sigma \oplus \nu\sigma$
	c	$L(\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-3/2}\sigma)$		$\sigma \oplus \nu^2\sigma$
	d	$\sigma 1_{\mathrm{GSp}(4)}$		—
V	a	$\delta([\xi, \nu\xi], \nu^{-1/2}\sigma)$		$\nu^2\sigma \oplus \xi\nu^2\sigma$
	b	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$		$\nu\sigma \oplus \xi\nu^2\sigma$
	c	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\xi\sigma)$		$\xi\nu\sigma \oplus \nu^2\sigma$
	d	$L(\nu\xi, \xi \rtimes \nu^{-1/2}\sigma)$		—
VI	a	$\tau(S, \nu^{-1/2}\sigma)$		$(\nu^2\sigma)[2]$
	b	$\tau(T, \nu^{-1/2}\sigma)$		—
	c	$L(\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$		$\nu\sigma$
	d	$L(\nu, 1_{F^\times} \rtimes \nu^{-1/2}\sigma)$		$(\nu\sigma)[2]$
VII		$\chi \rtimes \pi$		0
VIII	a	$\tau(S, \pi)$		0
	b	$\tau(T, \pi)$		0
IX	a	$\delta(\nu\xi, \nu^{-1/2}\pi(\mu))$		0
	b	$L(\nu\xi, \nu^{-1/2}\pi(\mu))$		—

		representation		$V_{N,T,\Lambda}$
X		$\pi \rtimes \sigma$	$\omega_\pi \neq 1$	$\nu^{3/2}\omega_\pi\sigma \oplus \nu^{3/2}\sigma$
			$\omega_\pi = 1$	$(\nu^{3/2}\sigma)[2]$
XI	a	$\delta(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$		$\nu^2\sigma$
	b	$L(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$		$\nu\sigma$
		supercuspidal		0

Using these facts, we can calculate, starting from the Table 3.1 of Jacquet modules, the Jacquet-Waldspurger modules in the split case for most of the irreducible, admissible, non-generic, non-supercuspidal representations of $\mathrm{GSp}(4, F)$. The results are listed in Table 3.4. In this table we assume that the representation of $\mathrm{GSp}(4, F)$ admits a split (Λ, β)-Bessel functional; an entry "—" indicates that no such Bessel functional exists. The following cases require special arguments:

<u>Type IIb</u>: Assume that (π, V) is a representation of type IIb, i.e., $\pi = \chi 1_{\mathrm{GL}(2)} \rtimes \sigma$ with $\chi^2 \neq \nu^{\pm 1}$ and $\chi \neq \nu^{\pm 3/2}$. By Table 3.1,

$$V_N = \sigma\chi 1_{\mathrm{GL}(2)} \otimes \nu^{3/2}\chi^2\sigma + \sigma\chi 1_{\mathrm{GL}(2)} \otimes \nu^{3/2}\sigma + (\chi^2\sigma \times \sigma) \otimes \nu\chi\sigma.$$

Assume first that $\chi^2 \neq 1$. In this case V_N is a direct sum, and we can calculate $V_{N,T,\Lambda}$ according to (3.3.12). Note that a split Bessel model exists only for $\Lambda = (\sigma\chi) \circ \mathrm{N}_{L/F}$. Hence each of the three summands contributes a one-dimensional component to $V_{N,T,\Lambda}$.

Now assume that $\chi^2 = 1$, so that V_N is given by

$$V_N = \big(\sigma\chi 1_{\mathrm{GL}(2)} \otimes \nu^{3/2}\sigma + \sigma\chi 1_{\mathrm{GL}(2)} \otimes \nu^{3/2}\sigma\big) \oplus (\chi^2\sigma \times \sigma) \otimes \nu\chi\sigma. \quad (3.3.13)$$

We will show that the term in parantheses is a direct sum of $\mathrm{GL}(2, F)$-modules by contradiction. Assume that the term in parantheses in (3.3.13) is not a direct sum as a $\mathrm{GL}(2, F)$-module, then it is isomorphic to

$$A \longmapsto (\chi\sigma)(\det(A)) \left[\begin{smallmatrix} 1 & v(\det(A)) \\ & 1 \end{smallmatrix}\right]. \tag{3.3.14}$$

It follows there exists $\bar{v}_1$, $\bar{v}_2 \in V_N$ such that $2 \cdot (\sigma\chi 1_{\mathrm{GL}(2)} \otimes \nu^{3/2}\sigma) = \langle \bar{v}_1, \bar{v}_2 \rangle$ such that

$$\begin{aligned} \pi(A)\bar{v}_2 =& (\sigma\chi \circ \det)(A) \cdot \bar{v}_2, \text{ and} \\ \pi(A)\bar{v}_1 =& (\sigma\chi \circ \det)(A) \cdot \bar{v}_1 + v(\det(A)) \cdot (\sigma\chi \circ \det)(A) \cdot \bar{v}_1. \end{aligned}$$

For any $x \in F^\times$,

$$\pi(\left[\begin{smallmatrix} x & \\ & x \end{smallmatrix}\right])\bar{v}_1 = \sigma^2\chi^2(x)\bar{v}_1. \tag{3.3.15}$$

The action (3.3.15) is a consequence of the fact $\left[\begin{smallmatrix} x & \\ & x \end{smallmatrix}\right] \in \mathrm{GL}(2, F)$ embedded in $\mathrm{GSp}(4)$ as a center element, so $\left[\begin{smallmatrix} x & \\ & x \end{smallmatrix}\right]$ acts on $\bar{v}_1$ by the central character. On the other hand,

$$\begin{aligned} \pi(\left[\begin{smallmatrix} x & \\ & x \end{smallmatrix}\right])\bar{v}_1 = \pi(\left[\begin{smallmatrix} x & \\ & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & \\ & x \end{smallmatrix}\right])\bar{v}_1 &= \pi(\left[\begin{smallmatrix} x & \\ & 1 \end{smallmatrix}\right])((\sigma\chi)(x)\bar{v}_1 + v(x)(\sigma\chi)(x)\bar{v}_2) \\ &= (\sigma\chi)(x)\pi(\left[\begin{smallmatrix} x & \\ & 1 \end{smallmatrix}\right])\bar{v}_1 + v(x)(\sigma\chi)(x)\pi(\left[\begin{smallmatrix} x & \\ & 1 \end{smallmatrix}\right])\bar{v}_2 \\ &= (\sigma\chi)(x)(\sigma\chi)(x)\bar{v}_1 + v(x)(\sigma\chi)(x)\bar{v}_2) + v(x)(\sigma\chi)^2(x)\bar{v}_2 \\ &= (\sigma\chi)^2(x)\bar{v}_1 + 2 \cdot v(x)(\sigma\chi)^2(x)\bar{v}_2, \end{aligned}$$

for every $x \in F^\times$. Comparing with (3.3.15), $2 \cdot v(x)(\sigma\chi)^2(x)\bar{v}_2 = 0$, for every $x \in F^\times$ which is a contradiction. Hence, as a $\mathrm{GL}(2, F)$-module, the term in parantheses in (3.3.13) is a direct sum.

Next, we need to prove the term in parantheses of (3.3.13) is not a direct sum of $\mathrm{GL}(2,F)\times F^\times$-modules. Indeed, we switch to the normalized Jacquet module $\tilde{V}_N$, as given in Table A.3 of [10]:

$$\tilde{V}_N = \big(\chi 1_{\mathrm{GL}(2)} \otimes \sigma + \chi 1_{\mathrm{GL}(2)} \otimes \sigma\big) \oplus (\chi\nu^{-1/2} \times \chi\nu^{-1/2}) \otimes \chi\nu^{-1/2}\sigma. \quad (3.3.16)$$

We have

$$\begin{aligned}\mathrm{Hom}_{\mathrm{GL}(2,F)\otimes F^\times}(\tilde{V}_N, \chi 1_{\mathrm{GL}(2)} \otimes \sigma) &\cong \mathrm{Hom}_P(V, \delta_P^{1/2}(\chi 1_{\mathrm{GL}(2)} \otimes \sigma)) \\ &\cong \mathrm{Hom}_{\mathrm{GSp}(4,F)}(V, \chi 1_{\mathrm{GL}(2)} \rtimes \sigma).\end{aligned}$$

By Schur's lemma, this last space is one-dimensional. It follows that the term in parantheses in (3.3.16) cannot be a direct sum of $\mathrm{GL}(2,F)\times F^\times$-modules. The same is then true for the term in parantheses in (3.3.13). Hence, as a $\mathrm{GL}(2,F)\times F^\times$-module, it is necessarily isomorphic to

$$(A,u) \longmapsto (\chi\sigma)(\det(A))(\nu^{3/2}\sigma)(u)\left[\begin{smallmatrix} 1 & v(u) \\ & 1 \end{smallmatrix}\right]. \quad (3.3.17)$$

Applying the functor $(\ldots)_{T,\Lambda}$ to (3.3.17) gives a two-dimensional representation. Hence, we get

$$V_{N,T,\Lambda} = 2\cdot\nu^{3/2}\sigma 1_{F^\times} \oplus \nu\chi\sigma 1_{F^\times}, \quad (3.3.18)$$

where $2 \cdot \nu^{3/2}\sigma 1_{F^\times}$ is isomorphic to

$$u \longmapsto (\nu^{3/2}\sigma)(u)\left[\begin{smallmatrix} 1 & v(u) \\ & 1 \end{smallmatrix}\right]. \tag{3.3.19}$$

Hence,

$$V_{N,T,\Lambda} = (\nu^{3/2}\sigma 1_{F^\times})[2] \oplus \nu\chi\sigma 1_{F^\times} \tag{3.3.20}$$

Type VId: Let (π, V) be a representation of type VId, i.e, $\pi = L(\nu, 1_{F^\times} \rtimes \nu^{-1/2}\sigma)$. In this case, Bessel models exist for $\Lambda = \sigma \circ N_{L/F}$. The Jacquet module V_N of V admits a filtration as follows

$$0 \subset V_2 \subset V_1 \subset V_N, \tag{3.3.21}$$

where $V_2 \cong V_N/V_1 \cong \sigma 1_{\mathrm{GL}(2)} \otimes \nu\sigma$, $V_N/V_2 \cong (\sigma\nu^{1/2} \times \sigma\nu^{-1/2}) \otimes \nu\sigma$ and $V_1/V_2 \cong \sigma \mathrm{St}_{\mathrm{GL}(2)} \otimes \nu\sigma$. We consider the short exact sequence

$$0 \longrightarrow V_2 \longrightarrow V_N \longrightarrow V_N/V_2 \longrightarrow 0$$

It is clear that the reducible principal series $(\tau, W) = \sigma\nu^{1/2} \times \sigma\nu^{-1/2}$ does not contain any element v such that

$$\tau(\left[\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right])v = \sigma(a)v,$$

for every $a \in F^\times$. Hence, $0 \subset V_2 \subset V_N$ satisfies the condition (3.2.34). As above, one can prove

$$0 \longrightarrow (V_2)_{T,\Lambda} \longrightarrow V_{N,T,\Lambda} \longrightarrow (V_N/V_2)_{T,\Lambda} \longrightarrow 0$$

But $\dim(V_2)_{T,\Lambda} = \dim(V_N/V_2)_{T,\Lambda} = 1$, so $V_{N,T,\Lambda}$ is 2 dimensional. By arguments as in Type IIb, $2 \cdot (1_{\mathrm{GL}(2)} \otimes \nu\sigma)$ is not direct sum as an $\mathrm{GL}(2) \times F^\times$-module. It follows $V_{N,T,\Lambda} = (\nu\sigma)[2]$.

ii) Generic cases:

- Assume that (π, V) is of types I, IIa, IIIa, IVa, Va, VII, VIIIa, IXa, X or XIa. It is easy to obtain the algebraic decompositions of types VII, VIIIa, IXa whose Jacquet modules are trivial, or the algebraic decompositions of types IVa, XIa whose Jacquet modules are of length one. Otherwise,

$$V_N = \sum_{i=1}^{n} \tau_i \otimes \chi_i \tag{3.3.22}$$

where $n \geq 2$ and the τ_i's are infinite dimensional, irreducible representations of $\mathrm{GL}(2, F)$. By Lemma 3.3.1,

$$V_{N,T,\Lambda} = \sum_{i=1}^{n} ((\tau_i)_{T,\Lambda} \otimes \chi_i). \tag{3.3.23}$$

To understand the algebraic decomposition of (3.3.23), we have to determine whether the sum in (3.3.22) is a direct sum as a $F^\times$-module. If the characters χ_i in (3.3.22) are pairwise different, i.e, the sum in (3.3.22) is direct, we simply apply Lemma 3.3.1 to obtain (3.3.23); this argument applies to a direct sum as types IIIa and Va. Regarding types I, IIa, and X, there may be a chance that some of the χ_i are not distinct. We make use of the algebraic decompositions of the twisted Jacquet modules V_{S,ψ_1} of these types appearing in Table 3.2. Let us define $V(S, H, \psi_1 \times \chi) = \langle \pi(hs)v - \chi(h)\psi_1(s)v : h \in H, s \in S \text{ and } v \in V \rangle$

and $V_{S,H,\psi_1\times\chi} = V/V(S,H,\psi_1\times\chi)$, where χ is a character of H. Since the actions of S_1 and $H = \mathrm{GL}(1,F)$ commute,

$$V_{S,H,\psi_1\times\chi} = (V_{S,\psi_1})_{H,\chi} = (V_{N,H,\chi})_{S_1,\psi_1}. \tag{3.3.24}$$

where $(-)_{H,\chi}$, $(-)_{S_1,\psi_1}$ and $(-)_{N,H,\chi}$ are defined similarly. By the algebraic decompositions of the twisted Jacquet modules V_{S,ψ_1} in Table 3.2 and the first equality in (3.3.24),

$$\dim V_{S,H,\psi_1\times\chi} \leq 1. \tag{3.3.25}$$

The equality holds if $\chi = \chi_i$, for some i. By Lemma 3.3.5, we assume that

$$V_{N,H,\chi} = \sum_{j=1}^{k} \tau_{i_j},$$

where $k > 1$ and $1 \leq i_j \leq n$. Since τ_{i_j} is generic, each $(\tau_{i_j})_{S_1,\psi_1}$ is one-dimensional. On the other hand, the functor $(-)_{S_1,\psi_1}$ is exact, so $V_{S,H,\psi_1\times\chi} = (V_{N,H,\chi})_{S_1,\psi_1}$ is of length $k > 1$, which is a contradiction with (3.3.25). Hence, $V_{N,H,\chi}$ is of length at most one. It follows the sum (3.3.22) is not a direct sum for non-pairwise distinct χ_i. On the other hand, we also have the following equation for the split T

$$V_{N,T,H,\Lambda\times\chi} = (V_{N,T,\Lambda})_{H,\chi} = (V_{N,H,\chi})_{T,\Lambda}, \tag{3.3.26}$$

where $V(N,T,H,\Lambda\times\chi) = \{\pi(tnh)v - \Lambda(t)\chi(h)v : t \in T, n \in N, \text{ and } h \in H\}$, and $V_{N,T,H,\Lambda\times\chi} = V/V(N,T,H,\Lambda\times\chi)$. Using this equation (3.3.26) and the exactness of split Jacquet modules of the category $\mathcal{C}$, we obtain the algebraic structure of split Jacquet-Waldspurger modules as in Table 3.4 in the case that the characters χ_i of H appearing in (3.3.22) are not pairwise distinct.

- Let (π, V) be a representation of type VIa, i.e, $\pi = \tau(S, \nu^{-1/2}\sigma)$. In this case, Bessel models exist for all Λ. The Jacquet module V_N of V admits a filtration as follows

$$0 \subset V_2 \subset V_1 \subset V_N, \tag{3.3.27}$$

where $V_2 \cong \sigma\mathrm{St}_{\mathrm{GL}(2)} \otimes \nu^2\sigma$, $V_1/V_2 \cong \sigma 1_{F^\times} \otimes \nu^2\sigma$, $V_1 \cong (\sigma\nu^{1/2} \times \sigma\nu^{-1/2}) \otimes \nu^2\sigma$ and $V_N/V_1 \cong \sigma\mathrm{St}_{\mathrm{GL}(2)} \otimes \nu^2\sigma$. We consider the short exact sequence

$$0 \longrightarrow V_2 \longrightarrow V_N \longrightarrow V_N/V_2 \longrightarrow 0.$$

We denote $I = V_N/V_2$. We can write $I = I_1 \oplus I_2$ as vector spaces, where $I_1 \cong \sigma 1_{F^\times} \otimes \nu^2\sigma$ and $I_2 \cong I/I_1 \cong \sigma\mathrm{St}_{\mathrm{GL}(2)} \otimes \nu^2\sigma$ admit the following actions:

  - Let $W_0 \in I_1$ such that $I_1 = \langle W_0 \rangle$, then

$$\pi([\,{}^{a}\;{}_{1}\,])W_0 = \sigma(a)W_0.$$

  - For any $U \in I_2$, there exist $c \in \mathbb{C}$ and $U_a \in I_2$ such that

$$\pi([\,{}^{a}\;{}_{1}\,])U = U_a + cW_0.$$

We claim that $0 \longrightarrow (V_2)_{T,\Lambda} \longrightarrow V_{N,T,\Lambda}$.

Case 1. $\Lambda \neq \sigma \circ \mathrm{N}_{L/F}$. Assume that $I = V_N/V_2$ does not satisfy the condition (3.2.34), i.e, there is a non-zero $W \in I$ such that

$$\pi([\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}])W = \Lambda_1(a)W.$$

We write $W = U + cW_0$ for some $U \in I_2$ and $c \in \mathbb{C}$. Then

$$\begin{aligned}
\pi([\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}])W = \Lambda_1(a)W \iff & \pi([\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}])U + c\sigma(a)W_0 = \Lambda_1(a)U + c\Lambda_1(a)W_0 \\
\iff & U_a + dW_0 + c\sigma(a)W_0 = \Lambda_1(a)U + c\Lambda_1(a)W_0 \\
\iff & U_a - \Lambda_1(a)U = (c\Lambda_1(a) - c\sigma(a) - d)W_0
\end{aligned}$$

for $U_a \in I_2$, $d \in \mathbb{C}$. Hence, $U_a - \Lambda_1(a)U = 0$, i.e, $U_a = \Lambda_1(a)U$. We have

$$\pi([\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}])U = \Lambda_1(a)U + cW_0$$

so

$$\pi([\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}])\bar{U} = \Lambda_1(a)\bar{U} \text{ in } I/I_1 \cong \sigma\mathrm{St}_{\mathrm{GL}(2)} \otimes \nu^2\sigma$$

Hence, $\bar{U} = 0$ in I/I_1. By our choice of U, $U = 0$, i.e, $W = cW_0$ where c is non-zero. But $\Lambda \neq \sigma \circ \mathrm{N}_{L/F}$, then $\Lambda_1(a) \neq \sigma(a)$ for some $a \in F^\times$. Then

$$\pi([\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}])W = c\sigma(a)W_0 \neq \Lambda_1(a)W_0 = \Lambda_1(a)W$$

which is a contradiction. It follows V_N/V_2 satisfies the condition (3.2.34).

By Lemma 3.2.6, the sequence

$$0 \longrightarrow (V_2)_{T,\Lambda} \longrightarrow V_{N,T,\Lambda} \longrightarrow (V_N/V_2)_{T,\Lambda} \longrightarrow 0$$

is exact. Now we need to calculate $(V_N/V_2)_{T,\Lambda}$. Let us consider

$$0 \longrightarrow I_1 \longrightarrow I = V_N/V_2 \longrightarrow I_2 \longrightarrow 0$$

Since $\Lambda \neq \sigma \circ \mathrm{N}_{L/F}$, $(I_1)_{T,\Lambda} = 0$, so $(V_N/V_2)_{T,\Lambda} \cong (I_2)_{T,\Lambda} = \nu^2\sigma$. Hence, $V_{N,T,\Lambda} = 2 \cdot \nu^2\sigma$.

<u>Case 2</u>. $\Lambda = \sigma \circ \mathrm{N}_{L/F}$. First, we will consider the case when σ is unramified. We have the exact sequence

$$0 \longrightarrow V_1 \longrightarrow V_N \longrightarrow V_N/V_1 \longrightarrow 0,$$

where $V_1 \cong (\sigma\nu^{1/2} \times \sigma\nu^{-1/2}) \otimes \nu^2\sigma$, $V_2 \cong V_N/V_1 \cong \sigma\mathrm{St}_{\mathrm{GL}(2)} \otimes \nu^2\sigma$ is irreducible, so $0 \subset V_1 \subset V_N$ satisfies the condition (3.2.34). By Lemma 3.2.4,

$$U \cap V_1(T,\Lambda) = U \cap V_N(T,\Lambda) \tag{3.3.28}$$

where $U \subset V_1$ consists of all $u \in V_1$ such that

$$\pi(\left[\begin{smallmatrix} a & \\ & 1 \end{smallmatrix}\right])u = \sigma(a)u, \text{ for all } a \in \mathfrak{o}^\times.$$

Hence, the composition

$$U/(U \cap V_1(T,\Lambda)) \xrightarrow{i} V_1/V_1(T,\Lambda) \longrightarrow V_N/V_N(T,\Lambda)$$

is injective. We claim that i is indeed an isomorphism. Let us denote $U_2 = U \cap V_2$. Since V_2 is irreducible, by Lemma 3.2.5, $U_2/(U_2 \cap V_2(T,\Lambda))$ is one dimensional. Let $f_2 \in U_2$ be such that $\{\bar{f_2}\}$ is a basis of $U_2/(U_2 \cap V_2(T,\Lambda))$. On the other hand, σ is unramified, so V_1 has a spherical vector f_0 which is not in V_2 by Proposition 2.1.2 of [13], so $f_0 \notin U_2$. But $f_0 \in U$, and $U =< U_2, f_0 >$ as a complex vector space.

Now we claim that $V_1(T,\Lambda) = V_2$. We have

$$V_2(T,\Lambda) \subset V_2 \cap V_1(T,\Lambda) \subset V_2.$$

Since $V_2/V_2(T,\Lambda) = (V_2)_{T,\Lambda}$ is one dimensional, we have either $V_2(T,\Lambda) = V_2 \cap V_1(T,\Lambda)$ or $V_1(T,\Lambda) = V_2$. Assume that $V_2(T,\Lambda) = V_2 \cap V_1(T,\Lambda)$. It follows

$$0 \longrightarrow (V_2)_{T,\Lambda} \longrightarrow (V_1)_{T,\Lambda} \to (V_1/V_2)_{T,\Lambda} \longrightarrow 0,$$

i.e, $(V_1)_{T,\Lambda}$ is two dimensional, which is a contradiction with Lemma 3.2.9. Hence, $V_1(T,\Lambda) = V_2$, which implies $U \cap V_1(T,\Lambda) = U \cap V_2 = U_2$. It follows $U/(U \cap V_1(T,\Lambda)) = U/U_2$ is a one dimensional vector space which has a basis $\{\bar{f_0}\}$. Hence, $\dim U/U \cap V_1(T,\Lambda) = \dim(V_1)_{T,\Lambda}$, i.e, i is an isomorphism which induces an exact sequence

$$0 \longrightarrow (V_1)_{T,\Lambda} \longrightarrow (V_N)_{T,\Lambda} \longrightarrow (V_N/V_1)_{T,\Lambda} \longrightarrow 0.$$

Then $\dim(V_N)_{T,\Lambda} = \dim(V_1)_{T,\Lambda} + \dim(V_N/V_1)_{T,\Lambda} = 2$ and $V_{N,T,\Lambda} = 2 \cdot \nu^2\sigma$.

Let us consider σ to be arbitrary. Our argument relies on the following fact: If (π, V) is a representation of $\mathrm{GSp}(4, F)$, then

$$(\sigma^{-1}V)_{N,T,\Lambda_{\sigma^{-1}}} \cong V_{N,T,\Lambda}. \tag{3.3.29}$$

The isomorphism is induced from $\sigma^{-1}V \to V$ such that $v \mapsto v$, where

$$V(N, T, \Lambda) = (\sigma^{-1}V)(N, T, \Lambda_{\sigma^{-1}}).$$

In fact, $v = (\sigma^{-1}\pi)(tn) - \Lambda_{\sigma^{-1}}(t)w = (\sigma^{-1} \circ \det)(t)(\pi(tn) - \Lambda(t)n) \in V(N, T, \Lambda)$, for any $v \in (\sigma^{-1}V)(N, T, \Lambda_{\sigma^{-1}})$.

In our case, $(\pi, V) = \tau(S, \nu^{-1/2}\sigma)$, $(\sigma^{-1}\pi, \sigma^{-1}V) \cong \tau(S, \nu^{-1/2})$ and $\Lambda_{\sigma^{-1}} = 1_{F^\times} \circ \mathrm{N}_{L/F}$. Since $1_{F^\times}$ is unramified, $(\sigma^{-1}V)_{N,T,\Lambda_{\sigma^{-1}}}$ is a two dimensional vector space as above arguments. Hence, $\dim V_{N,T,\Lambda} = 2$, i.e, $V_{N,T,\Lambda} = 2 \cdot \nu^2\sigma$ by (3.3.29).

Now we claim that $V_{N,T,\Lambda} = (\nu^2\sigma)[2]$ in both cases. We consider the short exact sequence

$$0 \longrightarrow V_2 \longrightarrow V_N \longrightarrow V_N/V_2 \longrightarrow 0. \tag{3.3.30}$$

We have the induced short exact sequence of (3.3.30) with respect to the functor $(-)_{S_1,\psi_1}$

$$0 \longrightarrow (V_2)_{S_1,\psi_1} \longrightarrow (V_N)_{S_1,\psi_1} = V_{S,\psi_1} \longrightarrow (V_N/V_2)_{S_1,\psi_1} \longrightarrow 0,$$

where $(V_2)_{S_1,\psi_1} = \nu^2\sigma$, $(V_N/V_2)_{S_1,\psi_1} = \nu^2\sigma$, and $V_{S,\psi_1} = (\nu^2\sigma)[2]$ as in

Table 3.2. As above, we also have the following short exact sequence of split Jacquet-Waldpurger modules

$$0 \longrightarrow (V_2)_{T,\Lambda} \longrightarrow (V_N)_{T,\Lambda} \longrightarrow (V_N/V_2)_{T,\Lambda} \longrightarrow 0.$$

Applying Lemma 3.3.5, similar to the proof of the split cases above, one can show that

$$V_{N,T,\Lambda} = (\nu^2\sigma)[2].$$

$\square$

Chapter 4

Zeta integrals and L-factors

4.1 Asymptotic behavior

In 4.1.1 we clarify the notion of "asymptotic function". Using our previous results on Jacquet-Waldspurger modules, we can calculate the asymptotic behavior of all Bessel functions of all representations; see Table 4.2. Simultaneously, we obtain the precise structure as an $F^\times$-module of the Jacquet-Waldspurger modules in the non-split case; see Table 4.1.

4.1.1 Asymptotic functions

Let $\mathcal{L}$ be the vector space of functions $f : F^\times \to \mathbb{C}$ with the following properties:

i) There exists an open-compact subgroup Γ of $F^\times$ such that $f(u\gamma) = f(u)$ for all $u \in F^\times$ and all $\gamma \in \Gamma$.

ii) $f(u) = 0$ for $v(u) \ll 0$.

Such f arise if we restrict Bessel functions on $\mathrm{GSp}(4, F)$ to the subgroup H.

Clearly $\mathcal{L}$ contains the Schwartz space $\mathcal{S}(F^\times)$, i.e., the space of locally constant, compactly supported functions $F^\times \to \mathbb{C}$. We may think of the quotient $\mathcal{L}/\mathcal{S}(F^\times)$ as a space of "asymptotic functions", in the sense that the image of some $f \in \mathcal{L}$ in this quotient is determined by the values $f(u)$ for $v(u) \gg 0$.

There is an action $\bar{\pi}$ of $F^\times$ on $\mathcal{L}$ given by translation: $(\bar{\pi}(x)f)(u) = f(ux)$ for $x, u \in F^\times$. This is a smooth action by the properties of the elements of $\mathcal{L}$. The action preserves the subspace $\mathcal{S}(F^\times)$, so that we get an action on the quotient $\mathcal{L}/\mathcal{S}(F^\times)$.

For the proof of the following lemma, we will use the formula

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k P(k) = 0, \qquad P \in \mathbb{C}[X],\ \deg(P) < n. \tag{4.1.1}$$

This formula follows by differentiating the identity $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$ repeatedly and setting $x = -1$.

Lemma 4.1.1. *Let $\beta \in \mathbb{C}^\times$. For a positive integer n, let $\mathcal{F}_n(\beta)$ be the space of functions $f : \mathbb{Z}_{\geq 0} \to \mathbb{C}$ satisfying*

$$\sum_{k=0}^{n} \binom{n}{k} (-\beta)^{n-k} f(m+k) = 0 \qquad \text{for all } m \geq 0. \tag{4.1.2}$$

Then $\dim \mathcal{F}_n(\beta) = n$, and a basis of $\mathcal{F}_n(\beta)$ is given by the functions

$$f_j(m) = m^j \beta^m, \qquad m \geq 0, \tag{4.1.3}$$

for $j = 0, \ldots, n-1$.

Proof. It is clear from (4.1.2) that any $f \in \mathcal{F}_n(\beta)$ is determined by the values

$f(0), \ldots, f(n-1)$. Hence $\dim \mathcal{F}_n(\beta) \leq n$, and we only need to show that the functions f_j lie in $\mathcal{F}_n(\beta)$ and are linearly independent. The fact that the functions f_j lie in $\mathcal{F}_n(\beta)$ follows from (4.1.1). It is easy to prove that they are linearly independent. □

Proposition 4.1.2. *Let $\mathcal{K}$ be an $F^\times$-invariant subspace of $\mathcal{L}$ which contains $\mathcal{S}(F^\times)$ with finite codimension n. Assume that, as an $F^\times$-module, the quotient $\mathcal{K}/\mathcal{S}(F^\times)$ is isomorphic to $\sigma[n]$; see Section 3.2.1, for some character σ of $F^\times$. Then there exist $f_0, \ldots, f_{n-1} \in \mathcal{K}$ with the following properties:*

i) The images of $f_0, \ldots, f_{n-1}$ in $\mathcal{K}/\mathcal{S}(F^\times)$ are a basis of the quotient space.

ii) f_j has asymptotic behavior

$$f_j(x) = v(x)^j \sigma(x) \qquad \text{for all } x \in F^\times \text{ with } v(x) \gg 0, \tag{4.1.4}$$

for all $j \in \{0, \ldots, n-1\}$.

Proof. It suffices to show that every $f \in \mathcal{K}$ has the asymptotic form

$$f(x) = \sum_{k=0}^{n-1} c_k\, v(x)^k \sigma(x) \qquad \text{for all } x \in F^\times \text{ with } v(x) \gg 0 \tag{4.1.5}$$

for some constants c_k. We have $\sigma[n](u) = \sigma(u)\mathrm{id}$ for $u \in \mathfrak{o}^\times$ on all of $\sigma[n]$. Hence, for a fixed unit $u \in \mathfrak{o}^\times$,

$$\bar{\pi}(u)f - \sigma(u)f \in \mathcal{S}(F^\times). \tag{4.1.6}$$

It follows that there exists a $j_0 \geq 0$ such that

$$f(u\varpi^{m+j_0}) = \sigma(u)f(\varpi^{m+j_0}) \qquad \text{for all } m \geq 0. \tag{4.1.7}$$

Since $\mathfrak{o}^\times$ is compact and both sides of (4.1.7) are locally constant, we may choose j_0 large enough so that (4.1.7) holds for *all* $u \in \mathfrak{o}^\times$.

Every vector in $\sigma[n]$ is annihilated by $(\sigma[n](\varpi) - \lambda\,\mathrm{id})^n$, where we abbreviate $\lambda = \sigma(\varpi)$. Hence

$$(\bar{\pi}(\varpi) - \lambda\,\mathrm{id})^n f \in \mathcal{S}(F^\times) \tag{4.1.8}$$

for all $f \in \mathcal{K}$, or

$$\sum_{k=0}^{n} \binom{n}{k} (-\lambda)^{n-k} \bar{\pi}(\varpi^k) f \in \mathcal{S}(F^\times). \tag{4.1.9}$$

It follows that there exists a $j_0 \geq 0$ such that

$$\sum_{k=0}^{n} \binom{n}{k} (-\lambda)^{n-k} f(\varpi^{m+k+j_0}) = 0 \qquad \text{for all } m \geq 0. \tag{4.1.10}$$

We may assume that the same j_0 works for both (4.1.7) and (4.1.10). Setting $h(m) := f(\varpi^{m+j_0})$, equation (4.1.10) reads

$$\sum_{k=0}^{n} \binom{n}{k} (-\lambda)^{n-k} h(m+k) = 0 \qquad \text{for all } m \geq 0. \tag{4.1.11}$$

By Lemma 4.1.1, there exist constants $d_0, \ldots, d_{n-1}$ such that

$$h(m) = \sum_{k=0}^{n-1} d_k m^k \lambda^m \qquad \text{for all } m \geq 0. \tag{4.1.12}$$

We can then also find constants $c_0, \ldots, c_{n-1}$ such that

$$h(m) = \sum_{k=0}^{n-1} c_k (m+j_0)^k \lambda^{m+j_0} \qquad \text{for all } m \geq 0. \tag{4.1.13}$$

(To get the c_k's from the d_k's, expand $m^k = ((m+j_0) - j_0)^k$ in (4.1.12).) For

$x \in F^\times$ with $v(x) \geq j_0$, write $x = u\varpi^j$ with $u \in \mathfrak{o}^\times$ and $j \geq j_0$. Then

$$\begin{aligned} f(x) &\overset{(4.1.7)}{=} \sigma(u) f(\varpi^j) \\ &\overset{(4.1.13)}{=} \sigma(u) \sum_{k=0}^{n-1} c_k j^k \lambda^j \\ &= \sum_{k=0}^{n-1} c_k \, v(x)^k \sigma(x). \end{aligned}$$

This concludes the proof. □

Corollary 4.1.3. *Let U be a finite-dimensional submodule of $\mathcal{L}/\mathcal{S}(F^\times)$. Then each σ-component of U is indecomposable.*

Proof. Let $\mathcal{K}$ be the pre-image of U under the projection $\mathcal{L} \to \mathcal{L}/\mathcal{S}(F^\times)$. Assume that there exists a σ for which U_σ is decomposable. Then U_σ contains a direct sum $\sigma[n] \oplus \sigma[n']$ with $n, n' > 0$. By Proposition 4.1.2, there exist two functions $f, f' \in \mathcal{K}$ such that the image of f in $U = \mathcal{K}/\mathcal{S}(F^\times)$ lies in $\sigma[n]$, the image of f' lies in $\sigma[n']$, and such that

$$f(x) = \sigma(x), \qquad f'(x) = \sigma(x) \qquad \text{for all } x \in F^\times \text{ with } v(x) \gg 0. \tag{4.1.14}$$

It follows from (4.1.14) that f and f' have the same image in $\mathcal{K}/\mathcal{S}(F^\times)$, a contradiction. □

4.1.2 Asymptotic behavior of Bessel functions in the non-split case

Let (π, V) be an irreducible, admissible representation of $\mathrm{GSp}(4, F)$. Assume that V is the (Λ, β)-Bessel model of π with respect to a character Λ of T. We associate with each Bessel function $B \in V$ the function $\varphi_B : F^\times \to \mathbb{C}$ defined by $\varphi_B(u) = B(diag(u, u, 1, 1))$. Let $\mathcal{K}$ be the space spanned by all functions φ_B.

Lemma 4.1.4. *$\mathcal{K}$ contains $\mathcal{S}(F^\times)$.*

Proof. This follows by the same arguments as in Lemma 4.1 of [5]. $\square$

An easy argument as in Proposition 4.7.2 of [3], or as in Proposition 3.1 of [5], shows that if $B \in V(N)$, then φ_B has compact support. It is also true, and equally easy to see, that

$$B \in V(N, T, \Lambda) \quad \Longrightarrow \quad \varphi_B \text{ has compact support in } F^\times.$$

It follows that the linear map $B \mapsto \varphi_B$ induces a surjection

$$V_{N,T,\Lambda} \longrightarrow \mathcal{K}/\mathcal{S}(F^\times). \tag{4.1.15}$$

Lemma 4.1.5. *Assume that the map* (4.1.15) *is an isomorphism. Then every σ-component of $V_{N,T,\Lambda}$ is indecomposable as an $F^\times$-module.*

Proof. The map (4.1.15) induces an isomorphism of the respective σ-components. Hence the assertion follows from Corollary 4.1.3. $\square$

Proposition 4.1.6. *Suppose we are in the non-split case. Then the map* (4.1.15) *is an isomorphism.*

Proof. See Theorem 4.9 of [5]. $\square$

Recall that in Table 3.3 we determined the semisimplifications of the Jacquet-Waldspurger modules for all irreducible, admissible representations. In the non-split case, we can now determine the precise algebraic structure of these modules.

Corollary 4.1.7. *The algebraic structure of the Jacquet-Waldspurger modules $V_{N,T,\Lambda}$ for all irreducible, admissible representations of* $\mathrm{GSp}(4,F)$ *is given in Table 4.1, under the assumption that the representation (π,V) admits a non-split (Λ,β)-Bessel functional. An entry "—" indicates that no such Bessel functional exists.*

Proof. By Proposition 4.1.6 and Lemma 4.1.5, every σ-component of $V_{N,T,\Lambda}$ is indecomposable. This information, together with the semisimplifications from Table 3.3, gives the precise structure. □

For type I, we have to distinguish various cases, depending on the regularity of the inducing character:

$$V_{N,T,\Lambda} = \begin{cases} \nu^{3/2}\chi_1\chi_2\sigma \oplus \nu^{3/2}\chi_1\sigma \oplus \nu^{3/2}\chi_2\sigma \oplus \nu^{3/2}\sigma & \text{if } \chi_1\chi_2, \chi_1, \chi_2, 1 \text{ are} \\ & \quad \text{pairwise different,} \\ \nu^{3/2}\chi^2\sigma \oplus (\nu^{3/2}\chi\sigma)[2] \oplus \nu^{3/2}\sigma & \text{if } \chi := \chi_1 = \chi_2 \neq 1,\ \chi^2 \neq 1, \\ (\nu^{3/2}\chi\sigma)[2] \oplus (\nu^{3/2}\sigma)[2] & \text{if } \chi := \chi_1 = \chi_2 \neq 1,\ \chi^2 = 1, \\ (\nu^{3/2}\chi\sigma)[2] \oplus (\nu^{3/2}\sigma)[2] & \text{if } \{\chi_1,\chi_2\} = \{\chi \neq 1, 1\} \\ (\nu^{3/2}\sigma)[4] & \text{if } \chi_1 = \chi_2 = 1. \end{cases} \tag{4.1.16}$$

Corollary 4.1.8. *Table 4.2 shows the asymptotic behavior of the Bessel functions $B(\mathrm{diag}(u,u,1,1))$ for all irreducible, admissible representations (π,V) of* $\mathrm{GSp}(4,F)$*, where B runs through a non-split (Λ,β)-Bessel model of π. An entry "—" indicates that no such Bessel model exists.*

Proof. By Proposition 4.1.6, the map (4.1.15) is an isomorphism. We can thus use Proposition 4.1.2, which translates the algebraic structure of $V_{N,T,\Lambda}$ given in

Table 4.1: Jacquet-Waldspurger modules $V_{N,T,\Lambda}$. It is assumed that L is a field, and that the representation of $\mathrm{GSp}(4,F)$ admits a (Λ,ψ_β)-Bessel functional. An entry "—" indicates that no non-split Bessel functional exists.

		representation		$V_{N,T,\Lambda}$
I		$\chi_1 \times \chi_2 \rtimes \sigma$		see (4.1.16)
II	a	$\chi\mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$	$\chi^2 \neq 1$	$\nu^2\chi\sigma \oplus \nu^{3/2}\chi^2\sigma \oplus \nu^{3/2}\sigma$
			$\chi^2 = 1$	$\nu^2\chi\sigma \oplus (\nu^{3/2}\sigma)[2]$
	b	$\chi 1_{\mathrm{GL}(2)} \rtimes \sigma$	$\chi^2 \neq 1$	$\nu\chi\sigma \oplus \nu^{3/2}\chi^2\sigma \oplus \nu^{3/2}\sigma$
			$\chi^2 = 1$	$\nu\chi\sigma \oplus (\nu^{3/2}\sigma)[2]$
III	a	$\chi \rtimes \sigma\mathrm{St}_{\mathrm{GSp}(2)}$		$\chi\nu^2\sigma \oplus \nu^2\sigma$
	b	$\chi \rtimes \sigma 1_{\mathrm{GSp}(2)}$		—
IV	a	$\sigma\mathrm{St}_{\mathrm{GSp}(4)}$		$\nu^3\sigma$
	b	$L(\nu^2, \nu^{-1}\sigma\mathrm{St}_{\mathrm{GSp}(2)})$		$\nu^3\sigma \oplus \nu\sigma$
	c	$L(\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-3/2}\sigma)$		—
	d	$\sigma 1_{\mathrm{GSp}(4)}$		—
V	a	$\delta([\xi, \nu\xi], \nu^{-1/2}\sigma)$		$\nu^2\sigma \oplus \xi\nu^2\sigma$
	b	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$		$\nu\sigma \oplus \xi\nu^2\sigma$
	c	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\xi\sigma)$		$\xi\nu\sigma \oplus \nu^2\sigma$
	d	$L(\nu\xi, \xi \rtimes \nu^{-1/2}\sigma)$		$\xi\nu\sigma \oplus \nu\sigma$
VI	a	$\tau(S, \nu^{-1/2}\sigma)$		$(\nu^2\sigma)[2]$
	b	$\tau(T, \nu^{-1/2}\sigma)$		$\nu^2\sigma$
	c	$L(\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$		—
	d	$L(\nu, 1_{F^\times} \rtimes \nu^{-1/2}\sigma)$		—
VII		$\chi \rtimes \pi$		0
VIII	a	$\tau(S, \pi)$		0
	b	$\tau(T, \pi)$		0
IX	a	$\delta(\nu\xi, \nu^{-1/2}\pi(\mu))$		0
	b	$L(\nu\xi, \nu^{-1/2}\pi(\mu))$		0

		representation		$V_{N,T,\Lambda}$
X		$\pi \rtimes \sigma$	$\omega_\pi \neq 1$	$\nu^{3/2}\omega_\pi\sigma \oplus \nu^{3/2}\sigma$
			$\omega_\pi = 1$	$(\nu^{3/2}\sigma)[2]$
XI	a	$\delta(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$		$\nu^2\sigma$
	b	$L(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$		$\nu\sigma$
		supercuspidal		0

Table 4.1 into the asymptotic behavior of Bessel functions. $\square$

Remark: This result is to be understood in the sense that all the constants given in Table 4.2 are necessary, i.e., for any choice of $C_1, C_2, \ldots$ there exists a Bessel function B such that $B(\mathrm{diag}(u, u, 1, 1))$ has the asymptotic behavior given by this choice of constants.

Again, for type I we have to distinguish various cases:

$$|u|^{-3/2}\, B(\mathrm{diag}(u, u, 1, 1)) \tag{4.1.17}$$
$$= \begin{cases} C_1(\chi_1\chi_2\sigma)(u) + C_2(\chi_1\sigma)(u) + C_3(\chi_2\sigma)(u) + C_4\sigma(u) & \text{if } \chi_1\chi_2, \chi_1, \chi_2, 1 \text{ are} \\ & \text{ pairwise different,} \\ C_1(\chi^2\sigma)(u) + (C_2 + C_3 v(u))(\chi\sigma)(u) + C_4\sigma(u) & \text{if } \chi := \chi_1 = \chi_2 \neq 1, \\ & \chi^2 \neq 1, \\ (C_1 + C_2 v(u))(\chi\sigma)(u) + (C_3 + C_4 v(u))\sigma(u) & \text{if } \chi := \chi_1 = \chi_2 \neq 1, \\ & \chi^2 = 1, \\ (C_1 + C_2 v(u))(\chi\sigma)(u) + (C_3 + C_4 v(u))\sigma(u) & \text{if } \{\chi_1, \chi_2\} = \\ & = \{\chi \neq 1, 1\}, \\ (C_1 + C_2 v(u) + C_3 v^2(u) + C_4 v^3(u))\sigma(u) & \text{if } \chi_1 = \chi_2 = 1. \end{cases}$$

Table 4.2: Asymptotic behavior of $B(\mathrm{diag}(u,u,1,1))$ in the non-split case. An entry "—" indicates that no non-split Bessel functional exists.

		representation		$\lvert u\rvert^{-3/2}\,B(\mathrm{diag}(u,u,1,1))$
I		$\chi_1\times\chi_2\rtimes\sigma$		see (4.1.17)
II	a	$\chi\mathrm{St}_{\mathrm{GL}(2)}\rtimes\sigma$	$\chi^2\neq1$	$C_1(\nu^{1/2}\chi\sigma)(u)+C_2(\chi^2\sigma)(u)+C_3\sigma(u)$
			$\chi^2=1$	$C_1(\nu^{1/2}\chi\sigma)(u)+(C_2+C_3v(u))\sigma(u)$
	b	$\chi1_{\mathrm{GL}(2)}\rtimes\sigma$	$\chi^2\neq1$	$C_1(\nu^{-1/2}\chi\sigma)(u)+C_2(\chi^2\sigma)(u)+C_3\sigma(u)$
			$\chi^2=1$	$C_1(\nu^{-1/2}\chi\sigma)(u)+(C_2+C_3v(u))\sigma(u)$
III	a	$\chi\rtimes\sigma\mathrm{St}_{\mathrm{GSp}(2)}$		$C_1(\nu^{1/2}\chi\sigma)(u)+C_2(\nu^{1/2}\sigma)(u)$
	b	$\chi\rtimes\sigma1_{\mathrm{GSp}(2)}$		—
IV	a	$\sigma\mathrm{St}_{\mathrm{GSp}(4)}$		$C(\nu^{3/2}\sigma)(u)$
	b	$L(\nu^2,\nu^{-1}\sigma\mathrm{St}_{\mathrm{GSp}(2)})$		$C_1(\nu^{3/2}\sigma)(u)+C_2(\nu^{-1/2}\sigma)(u)$
	c	$L(\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)},\nu^{-3/2}\sigma)$		—
	d	$\sigma1_{\mathrm{GSp}(4)}$		—
V	a	$\delta([\xi,\nu\xi],\nu^{-1/2}\sigma)$		$C_1(\nu^{1/2}\xi\sigma)(u)+C_2(\nu^{1/2}\sigma)(u)$
	b	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma)$		$C_1(\nu^{1/2}\xi\sigma)(u)+C_2(\nu^{-1/2}\sigma)(u)$
	c	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\xi\sigma)$		$C_1(\nu^{-1/2}\xi\sigma)(u)+C_2(\nu^{1/2}\sigma)(u)$
	d	$L(\nu\xi,\xi\rtimes\nu^{-1/2}\sigma)$		$C_1(\nu^{-1/2}\xi\sigma)(u)+C_2(\nu^{-1/2}\sigma)(u)$
VI	a	$\tau(S,\nu^{-1/2}\sigma)$		$(C_1+C_2v(u))(\nu^{1/2}\sigma)(u)$
	b	$\tau(T,\nu^{-1/2}\sigma)$		$C(\nu^{1/2}\sigma)(u)$
	c	$L(\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma)$		—
	d	$L(\nu,1_{F^\times}\rtimes\nu^{-1/2}\sigma)$		—
VII		$\chi\rtimes\pi$		0
VIII	a	$\tau(S,\pi)$		0
	b	$\tau(T,\pi)$		0
IX	a	$\delta(\nu\xi,\nu^{-1/2}\pi(\mu))$		0
	b	$L(\nu\xi,\nu^{-1/2}\pi(\mu))$		0

	representation		$\|u\|^{-3/2}\, B(\mathrm{diag}(u,u,1,1))$
X	$\pi \rtimes \sigma$	$\omega_\pi \neq 1$	$C_1(\omega_\pi\sigma)(u) + C_2\sigma(u)$
		$\omega_\pi = 1$	$(C_1 + C_2 v(u))\sigma(u)$
XI a	$\delta(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$		$C(\nu^{1/2}\sigma)(u)$
b	$L(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$		$C(\nu^{-1/2}\sigma)(u)$
	supercuspidal		0

Remark 4.1.9. The proof of Proposition 4.1.6 given in [5] is based on the exactness of the Waldspurger functor, which is only true in the non-split case. Assume that (π, V) is an irreducible, admissible representation of $\mathrm{GSp}(4,F)$ which admits a *split* Bessel model $\mathcal{B}(\pi, \Lambda, \beta)$. Then we still have the surjection (4.1.15), which implies that the space of asymptotic functions $\mathcal{K}/\mathcal{S}(F^\times)$, as an $F^\times$-module, is a quotient of the Jacquet-Waldspurger module $V_{N,T,\Lambda}$. Starting from the $V_{N,T,\Lambda}$ given in Table 3.1, the $V_{N,T,\Lambda}$ can be calculated in many cases, but some of them pose difficulties, again due to the fact that the Waldspurger functor in the split case is not exact. Thus, complete results in the split case would follow from controling the kernel of the map (4.1.15).

The current methods still allow for some preliminary results on the asymptotic behavior of the functions $B(\mathrm{diag}(u,u,1,1))$ in the split case. More precisely, it is not difficult to create a table similar to Table 4.2, but it is unclear if all the constants C_i in such a table are really necessary. What is clear is that every $B(\mathrm{diag}(u,u,1,1))$ is of the general form

$$B(\mathrm{diag}(u,u,1,1)) = \sum_{i=1}^{n} C_i v(u)^{k_i} \sigma_i(u) \qquad \text{for } v(u) \gg 0 \tag{4.1.18}$$

with k_i non-negative integers, σ_i characters of $F^\times$, and $C_i \in \mathbb{C}$.

4.1.3 Asymptotic behavior of non-generic Bessel functions in the split case

Let (π, V) be an irreducible, admissible representation of $\mathrm{GSp}(4, F)$. Assume that V is the split (Λ, β)-Bessel model of π with respect to a character Λ of T. Recall that we choose

$$\beta = \begin{bmatrix} & 1/2 \\ 1/2 & \end{bmatrix}.$$

We associate with each Bessel function $B \in V$ the function $\varphi_B : F^\times \to \mathbb{C}$ defined by $\varphi_B(u) = B(\mathrm{diag}(u, u, 1, 1))$. Let $\mathcal{K}(\pi)$ be the space spanned by all functions φ_B.

Lemma 4.1.10. $\{\varphi_B : u \mapsto B(\mathrm{diag}(u, u, 1, 1)) : B \in V(N, T, \Lambda)\} = \mathcal{S}(F^\times)$.

Proof. Let $\mathcal{K}'$ be the space of functions $\varphi :\ F^\times \to \mathbb{C}$ of the form

$$\varphi_B(x) = B(\begin{bmatrix} x &&& \\ & x && \\ && 1 & \\ &&& 1 \end{bmatrix}), \qquad x \in F^\times, \tag{4.1.19}$$

where B runs through $V(N, T, \Lambda)$. An easy argument as in Proposition 4.7.2 of [3] shows that $\mathcal{K}' \subset \mathcal{S}(F^\times)$.

We choose $B \in V$ with $B(1) \neq 0$, and choose a symmetric 2×2-matrix X such that $\psi(\mathrm{tr}(\beta X)) \neq 1$. Then

$$B' := \pi([\begin{smallmatrix} 1 & X \\ & 1 \end{smallmatrix}])B - B \in V(N) \subset V(N, T, \Lambda),$$

and

$$\varphi_{B'}(1) = B'(1) = B([\begin{smallmatrix} 1 & X \\ & 1 \end{smallmatrix}]) - B(1) = (\psi(\mathrm{tr}(\beta X)) - 1)B(1) \neq 0.$$

This proves $\mathcal{K}' \neq 0$.

A straightforward calculation shows that the following holds for any $y \in F^\times$ and any symmetric 2×2-matrix X:

$$\text{If } B' = \pi(\begin{bmatrix} y & & & \\ & y & & \\ & & 1 & \\ & & & 1 \end{bmatrix})B, \text{ then } \varphi_{B'}(u) = \varphi_B(uy). \tag{4.1.20}$$

$$\text{If } B' = \pi([\begin{smallmatrix} 1 & X \\ & 1 \end{smallmatrix}])B, \text{ then } \varphi_{B'}(u) = \psi(u\,\mathrm{tr}(\beta X))\varphi_B(u). \tag{4.1.21}$$

Note that if $B \in V(N,T,\Lambda)$, then the functions B' defined in (4.1.20) and (4.1.21) are also in $V(N,T,\Lambda)$. By Proposition 4.7.3 of [3], $\mathcal{K}' = \mathcal{S}(F^\times)$. $\square$

In this section, we determine the dimension and an explicit basis of $\mathcal{K}(\pi)/\mathcal{S}(F^\times)$. By Lemma 4.1.10, we have a surjection

$$V/V(N,T,\Lambda) \longrightarrow \mathcal{K}(\pi)/\mathcal{S}(F^\times). \tag{4.1.22}$$

From (4.1.22) it follows that

$$\dim V_{N,T,\Lambda} \geq \dim \mathcal{K}(\pi)/\mathcal{S}(F^\times). \tag{4.1.23}$$

We can get precise results for the asymptotic behavior in the split case for those representations for which the Jacquet-Waldspurger module is known by Table 3.4, *and* for which there is equality in (4.1.23). In fact, this holds in the non-split case. In the split case, we will show that the equality in (4.1.23) is also true for all non-generic representations of $\mathrm{GSp}(4,F)$ and generic representations except finitely many choices of the characters Λ of T. We denote

$$N_0 = \{\begin{bmatrix} 1 & & x & \\ & 1 & & y \\ & & 1 & \\ & & & 1 \end{bmatrix} : x, y \in F\},\ N' = \{\begin{bmatrix} 1 & & & z \\ & 1 & z & \\ & & 1 & \\ & & & 1 \end{bmatrix} : z \in F\}, \tag{4.1.24}$$

$$N_0(m) = \{\begin{bmatrix} 1 & & x & \\ & 1 & & y \\ & & 1 & \\ & & & 1 \end{bmatrix} : x, y \in \mathfrak{p}^m\}, \text{ and } N'(m) = \{\begin{bmatrix} 1 & & & z \\ & 1 & z & \\ & & 1 & \\ & & & 1 \end{bmatrix} : z \in \mathfrak{p}^m\}. \tag{4.1.25}$$

Let $N(m) = N_0(m)N'(m)$. It is obvious that $N = N_0 N'$.

Lemma 4.1.11. *Let (π, V) be an irreducible, admissible infinite-dimensional representation of* $\mathrm{GSp}(4, F)$*, then* $V^{N'} = \{0\}$.

Proof. Let $v \in V^{N'}$. Since π is smooth, there exists a $g = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}(2, F) - B_2$ such that

$$\pi(\begin{bmatrix} a & & & b \\ & a & b & \\ & c & d & \\ c & & & d \end{bmatrix})v = v. \tag{4.1.26}$$

where B_2 is the standard Borel of $\mathrm{SL}(2, F)$. But $\mathrm{SL}(2)$ is generated by the unipotent radical and an element of $\mathrm{SL}(2, F) - B_2$ and N' is isomorphic to the canonical unipotent radical of B_2. Hence,

$$\pi(\begin{bmatrix} a & & & b \\ & a & b & \\ & c & d & \\ c & & & d \end{bmatrix})v = v, \text{ for any } \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}(2, F).$$

For any $x, y \in F$, we choose a large enough n such that v is fixed by

$$\begin{bmatrix} 1 & & \varpi^{2n}x & \\ & 1 & & \varpi^{2n}y \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

On the other hand, $\begin{bmatrix} \varpi^{\pm n} & & & \\ & \varpi^{\pm n} & & \\ & & \varpi^{\mp n} & \\ & & & \varpi^{\mp n} \end{bmatrix}$ is of the form (4.1.26). Hence,

$$\pi(\begin{bmatrix} 1 & & x & \\ & 1 & & y \\ & & 1 & \\ & & & 1 \end{bmatrix})v = \pi(\begin{bmatrix} \varpi^{-n} & & & \\ & \varpi^{-n} & & \\ & & \varpi^{n} & \\ & & & \varpi^{n} \end{bmatrix}\begin{bmatrix} 1 & & \varpi^{2n}x & \\ & 1 & & \varpi^{2n}y \\ & & 1 & \\ & & & 1 \end{bmatrix}\begin{bmatrix} \varpi^{n} & & & \\ & \varpi^{n} & & \\ & & \varpi^{-n} & \\ & & & \varpi^{-n} \end{bmatrix})v = v.$$

It follows v is fixed by N_0, so v is fixed by $N = N_0 N'$. Similar to the proof of Proposition 4.6 of [5], one can prove that $v = 0$. $\square$

We consider the exact sequence

$$0 \longrightarrow V(T,\Lambda)/(V(T,\Lambda) \cap V(N,\psi)) \longrightarrow V_{N,\psi} \longrightarrow V_{N,T,\Lambda\otimes\psi} \longrightarrow 0. \tag{4.1.27}$$

The exactness holds by the fact that $V(N,T,\Lambda \otimes \psi) = V(T,\Lambda) + V(N,\psi)$ and $V(N,T,\Lambda \otimes \psi)/V(N,\psi) = V(T,\Lambda)/(V(T,\Lambda) \cap V(N,\psi))$. Let us consider the twisted Jacquet modules $V_{N,\psi}$, for which there are four possibilities

$$\dim V_{N,\psi} \in \{0, 1, 2, +\infty\}. \tag{4.1.28}$$

If $\dim V_{N,\psi} = 0$, i.e, the Bessel models do not exist, $V = V(N,\psi)$, then $V(N,\psi) = V(N,T,\Lambda \otimes \psi)$.

If $\dim V_{N,\psi} = 1$, then $V(T,\Lambda)/(V(T,\Lambda) \cap V(N,\psi)) = 0$ by the exactness of (4.1.27) and the fact that $\dim V/V(N,T,\Lambda \otimes \psi) = 1$ which is the consequence of

$$\mathrm{Hom}_R(V, \mathbb{C}_{\Lambda\otimes\psi}) \cong \mathrm{Hom}_{\mathbb{C}}(V_{N,T,\Lambda\otimes\psi}, \mathbb{C}).$$

Hence, $V(T,\Lambda) \subset V(N,\psi)$, i.e, $V(N,\psi) = V(N,T,\Lambda \otimes \psi)$.

Lemma 4.1.12. *If $V_{N,\psi}$ is one dimensional, then $\varphi_v \equiv 0$ if and only if $v \in V(N_0,T,\Lambda)$.*

Proof. Assuming that $v \in V(N_0,T,\Lambda)$, it is easy to check that $\varphi_v \equiv 0$ by straightforward calculations. Conversely, let $v \in V$ be such that $\varphi_v \equiv 0$, i.e, $\varphi_v(x) = 0$ for every $x \in F^\times$. Hence, as we are in the case that $\dim V_{N,\psi} = 1$,

$$\pi(\begin{bmatrix} x & & & \\ & x & & \\ & & 1 & \\ & & & 1 \end{bmatrix})v \in V(N,T,\Lambda \otimes \psi) = V(N,\psi)$$

$$\iff \int_{N(-m)} \psi(-n)\pi(n)\big[\pi(\begin{bmatrix} x &&& \\ & x && \\ && 1 & \\ &&& 1 \end{bmatrix})v\big]\, dn = 0, \text{ for large enough } m.$$

Similar to the calculations in Lemma 5.3 in [15], there exists a large enough $M(x)$ which depends on x such that

$$\int_{\mathfrak{p}^{-m}} \psi(\begin{bmatrix} 1 &&& xz \\ & 1 & xz & \\ && 1 & \\ &&& 1 \end{bmatrix})\pi(\begin{bmatrix} 1 &&& z \\ & 1 & z & \\ && 1 & \\ &&& 1 \end{bmatrix})\big[\int_{N_0(-m)} \pi(n_0)v\, dn_0\big]\, dz = 0,$$

for every $m \geq M(x)$. By Lemma 5.2 of [15],

$$\int_{N_0(-m)} \pi(n_0)v\, dn_0 \in V^{N'}.$$

By Lemma 4.1.11,

$$\int_{N_0(-m)} \pi(n_0)v\, dn_0 = 0.$$

It follows $v \in V(N_0) \subset V(N_0, T, \Lambda)$. $\square$

Theorem 4.1.13. *If $V_{N,\psi}$ is one dimensional, then $\varphi_v \in \mathcal{S}(F^\times)$ if and only if $v \in V(N, T, \Lambda)$.*

Proof. This is a consequence of Lemma 4.1.10 and Lemma 4.1.12 $\square$

Theorem 4.1.14. *If $V_{N,\psi}$ is two dimensional, then $\varphi_v = 0$ if and only if $v \in V(N_0, T, \Lambda)$.*

Proof. ($\Longleftarrow$). It is obvious.

($\Longrightarrow$). If $\dim(V_{N,\psi}) = 2$, we are in the case of types IIIb and IVc. By Lemma 5.3.4(i) of [11],

$$V = \mathbb{C}w_1 + \mathbb{C}w_2 + V(N, \psi),\ V_{N,\psi} = \mathbb{C}\bar{w}_1 \oplus \mathbb{C}\bar{w}_2, \tag{4.1.29}$$

$$\pi(\begin{bmatrix} a & & & \\ & b & & \\ & & b & \\ & & & a \end{bmatrix})\bar{w}_1 = \chi_1(a)\chi_2(b)\bar{w}_1, \text{ and } \pi(\begin{bmatrix} a & & & \\ & b & & \\ & & b & \\ & & & a \end{bmatrix})\bar{w}_2 = \chi_1(b)\chi_2(a)\bar{w}_2, \quad (4.1.30)$$

where $\bar{w}_1$, $\bar{w}_2$ are projections of w_1, w_2, respectively in $V_{N,\psi}$, χ_1, χ_2 are some characters of $F^\times$ and $\chi_1 \neq \chi_2$. In particular, Bessel models exist if $\Lambda = (\chi_1, \chi_2)$ or $\Lambda = (\chi_2, \chi_1)$. Without loss of generality, we may assume $\Lambda = (\chi_1, \chi_2)$. Let us denote $\Lambda' := (\chi_2, \chi_1)$, so $V_{N,T,\Lambda\otimes\psi} = \mathbb{C}\bar{\bar{w}}_1$ and $V_{N,T,\Lambda'\otimes\psi} = \mathbb{C}\bar{\bar{w}}_2$, where $\bar{\bar{w}}_1$, $\bar{\bar{w}}_2$ are projections of w_1, w_2 in $V_{N,T,\Lambda\otimes\psi}$, $V_{N,T,\Lambda'\otimes\psi}$.

We observe that $V = V(T,\Lambda) + V(T,\Lambda')$. Hence, we may assume $w_2 \in V(T,\Lambda)$ and $w_1 \in V(T,\Lambda')$ such that (4.1.29) holds. Notice that $\bar{w}_1$ and $\bar{w}_2$ may not satisfy the equations (4.1.30) by our assumption. But we still have

$$\pi(\begin{bmatrix} a & & & \\ & b & & \\ & & b & \\ & & & a \end{bmatrix})\bar{\bar{w}}_1 = \chi_1(a)\chi_2(b)\bar{\bar{w}}_1, \text{ and } \pi(\begin{bmatrix} a & & & \\ & b & & \\ & & b & \\ & & & a \end{bmatrix})\bar{\bar{w}}_2 = \chi_1(b)\chi_2(a)\bar{\bar{w}}_2, \quad (4.1.31)$$

We claim that

$$V(N,\psi) = V(N,\psi) \cap V(T,\Lambda) + V(N,\psi) \cap V(T,\Lambda'). \quad (4.1.32)$$

It is clear that "$\supseteq$" holds. On the other hand, for any $v \in V(N,\psi)$, $v = v_1 + v_2$, where $v_1 \in V(T,\Lambda)$, $v_2 \in V(T,\Lambda')$. By (4.1.29), $v_1 = c_1 w_1 + c_2 w_2 + z_1$, for some constants $c_1, c_2 \in \mathbb{C}$ and $z_1 \in V(N,\psi)$. Hence, $c_1 w_1 + z_1 = v_1 - c_2 w_2 \in V(T,\Lambda)$. It follows

$$c_1\bar{\bar{w}}_1 = c_1\bar{\bar{w}}_1 + \bar{\bar{z}}_1 = \overline{\overline{c_1 w_1 + z_1}} = 0 \text{ in } V_{N,T,\Lambda\otimes\psi},$$

so $c_1 = 0$, i.e, $v_1 = c_2 w_2 + z_1$ or $z_1 = v_1 - c_2 w_2 \in V(T,\Lambda)$, i.e, $z_1 \in V(N,\psi) \cap V(T,\Lambda)$. Similarly, one can prove $v_2 = d_1 w_1 + z_2$, where $z_2 \in V(N,\psi) \cap V(T,\Lambda')$.

But $v = v_1 + v_2 = c_2 w_2 + z_1 + d_1 w_1 + z_2 \in V(N, \psi)$. It follows

$$c_2 \bar{w}_2 + d_1 \bar{w}_1 = c_2 \bar{w}_2 + \bar{z}_1 + d_1 \bar{w}_1 + \bar{z}_2 = \overline{c_2 w_2 + z_1 + d_1 w_1 + z_2} = \bar{v} = 0 \text{ in } V_{N,\psi}.$$

Since $\{\bar{w}_1, \bar{w}_2\}$ is a basis of $V_{N,\psi}$, $c_2 = d_1 = 0$, i.e, $v = v_1 + v_2 \in (V(N, \psi) \cap V(T, \Lambda)) + (V(N, \psi) \cap V(T, \Lambda'))$. Hence, (4.1.32) holds.

Next, we will show that

$$V(T, \Lambda') = \mathbb{C} w_1 + V(T, \Lambda') \cap V(N, \psi). \tag{4.1.33}$$

It is clear "$\supseteq$" holds. On the other hand, let $v \in V(T, \Lambda')$, $v = a_1 w_1 + a_2 w_2 + z$, where $z \in V(N, \psi)$ and $a_1, a_2 \in \mathbb{C}$. Hence, $a_2 w_2 + z = v - a_1 w_1 \in V(T, \Lambda')$. It follows

$$a_2 \bar{\bar{w}}_2 = a_2 \bar{\bar{w}}_2 + \bar{\bar{z}} = \overline{\overline{a_2 w_2 + z}} = 0 \text{ in } V_{N,T,\Lambda' \otimes \psi}.$$

Hence, $a_2 = 0$, i.e, $v = a_1 w_1 + z$. Hence, $z = v - a_1 w_1 \in V(T, \Lambda')$, i.e, $z \in V(T, \Lambda') \cap V(N, \psi)$. It follows the equality (4.1.33) holds.

Let $u \in V$ such that

$$B_u\left(\begin{bmatrix} x & & & \\ & x & & \\ & & 1 & \\ & & & 1 \end{bmatrix}\right) = 0, \text{ for every } x \in F^\times.$$

In particular, if $x = 1$, $u \in V(N, T, \Lambda \otimes \psi)$. By (4.1.32), u can be written as

$$u = u_1 + u_2,$$

where $u_1 \in V(T, \Lambda)$ and $u_2 \in V(T, \Lambda') \cap V(N, \psi)$. In fact, $u \in V(N, T, \Lambda \otimes \psi)$, so $u = u_1' + u_2'$, where $u_1' \in V(T, \Lambda)$, $u_2' \in V(N, \psi)$. By (4.1.32), $u_2' = u"_1 + u"_2$, where $u"_1 \in V(N,\psi) \cap V(T,\Lambda) \subset V(T,\Lambda)$, $u"_2 \in V(N,\psi) \cap V(T,\Lambda')$. Hence, we pick $u_1 = u_1' + u"_1, u_2 = u"_2$.

Besides, for any $x \in F^\times$,

$$B_{u_2}(\begin{bmatrix} x &&& \\ & x && \\ && 1 & \\ &&& 1 \end{bmatrix}) = B_u(\begin{bmatrix} x &&& \\ & x && \\ && 1 & \\ &&& 1 \end{bmatrix}) - B_{u_1}(\begin{bmatrix} x &&& \\ & x && \\ && 1 & \\ &&& 1 \end{bmatrix}) = 0.$$

Since $u_2 \in V(T,\Lambda')$, $\pi(\begin{bmatrix} x &&& \\ & x && \\ && 1 & \\ &&& 1 \end{bmatrix})u_2 \in V(T,\Lambda')$. By (4.1.33),

$$\pi(\begin{bmatrix} x &&& \\ & x && \\ && 1 & \\ &&& 1 \end{bmatrix})u_2 = cw_1 + w_x,$$

for some $w_x \in V(T,\Lambda') \cap V(N,\psi)$. But

$$0 = B_{u_2}(\begin{bmatrix} x &&& \\ & x && \\ && 1 & \\ &&& 1 \end{bmatrix}) = B_{\pi(\begin{bmatrix} x &&& \\ & x && \\ && 1 & \\ &&& 1 \end{bmatrix})u_2}(1) = cB_{w_1}(1) + B_{w_x}(1) = cB_{w_1}(1).$$

Since $V_{N,T,\Lambda\otimes\psi} = \mathbb{C}\bar{\bar{w}}_1$, $B_{w_1}(1) \neq 0$. Hence, $c = 0$, i.e,

$$\pi(\begin{bmatrix} x &&& \\ & x && \\ && 1 & \\ &&& 1 \end{bmatrix})u_2 = w_x \in V(T,\Lambda') \cap V(N,\psi) \subset V(N,\psi),$$

for every $x \in F^\times$. Similar to the technique of integration as shown in the proof of Lemma 4.1.12, one can prove $u_2 \in V(N_0)$. It follows $u = u_1 + u_2 \in V(N_0, T, \Lambda)$.

□

By Lemma 4.1.10 and Theorem 4.1.14, we obtain the following consequences

Corollary 4.1.15. *If $V_{N,\psi}$ is two dimensional, then $\varphi_v \in \mathcal{S}(F^\times)$ if and only if $v \in V(N,T,\Lambda)$.*

We have the following results for the isomorphism (4.1.22) in the split case.

Notice that Bessel models do not exist for types IVd, Vd, VIb, VIIIb and IXb in the split case.

Corollary 4.1.16. *Let (π, V) be a non-generic representation of* $\mathrm{GSp}(4, F)$*. Assume that we are in the split case, and π has a (Λ, β)-Bessel model. Then $V_{N,T,\Lambda} \cong \mathcal{K}(\pi)/\mathcal{S}(F^\times)$ as vector spaces.*

Proof. If $V_N = 0$, i.e, π is of types VIIIb, IXb, it is obvious that $V_{N,T,\Lambda} \cong \mathcal{K}(\pi)/\mathcal{S}(F^\times) = 0$ using inequality (4.1.23).

If (π, V) is of types IIb, IVb, Vb, Vc, VIc, VId, XIb, $V_{N,T,\Lambda\otimes\psi}$ is one dimensional which is a consequence of the Table A.6 of [10]. Hence, the isomorphism $V_{N,T,\Lambda\otimes\psi} \cong \mathcal{K}(\pi)/\mathcal{S}(F^\times)$ is an easy implication of Theorem 4.1.13.

On the other hand, if (π, V) is of types IIIb and IVc, $V_{N,T,\Lambda\otimes\psi}$ is two dimensional. By Corollary 4.1.15, $V_{N,T,\Lambda} \cong \mathcal{K}(\pi)/\mathcal{S}(F^\times)$. $\square$

4.1.4 Asymptotic behavior of generic Bessel functions in the split case

We treat the generic case by determining the algebraic decomposition of the module $V_{N_0,T,\Lambda}$. In fact, $V_{N_0,T,\Lambda}$ is a HN'-module, where

$$N' = \{ \begin{bmatrix} 1 & & & z \\ & 1 & z & \\ & & 1 & \\ & & & 1 \end{bmatrix} : z \in F\},$$

and HN' is isomorphic to the mirabolic group

$$M_2 = \begin{bmatrix} * & * \\ & 1 \end{bmatrix} \cap \mathrm{GL}(2, F). \tag{4.1.34}$$

By Theorem 8.3 of [4],

$$V_{N_0,T,\Lambda}(N') \cong \text{c-Ind}_{N'}^{HN'}(V_{N_0,T,\Lambda})_{N',\psi} \cong \text{c-Ind}_{N'}^{HN'} V_{N,T,\Lambda\otimes\psi}. \tag{4.1.35}$$

By the uniqueness of Bessel models, $V_{N,T,\Lambda\otimes\psi}$ is one-dimensional. As a N'-module, $V_{N,T,\Lambda\otimes\psi} \cong \psi$. By Corollary 8.2 of [4],

$$V_{N_0,T,\Lambda}(N') \cong \text{c-Ind}_{N'}^{HN'} \psi \tag{4.1.36}$$

is irreducible as HN'-modules. We have the filtration

$$0 \subset V_{N_0,T,\Lambda}(N') \subset V_{N_0,T,\Lambda}. \tag{4.1.37}$$

On the other hand, $V_{N,T,\Lambda} \cong V_{N_0,T,\Lambda}/V_{N_0,T,\Lambda}(N')$ gives us a composition series of $V_{N_0,T,\Lambda}$

$$0 \subset V_0 \subset V_1 \subset \cdots \subset V_n = V_{N_0,T,\Lambda}, \tag{4.1.38}$$

where $V_0 \cong V_{N_0,T,\Lambda}(N')$, $V_i/V_{i-1} \cong \chi_i$, for $1 \le i \le n$, and

$$V_{N,T,\Lambda} = \sum_{i=1}^{n} \chi_i$$

is the decomposition of $V_{N,T,\Lambda}$ in Table 3.4. It follows the semiplification of $V_{N_0,T,\Lambda}$

$$V_{N_0,T,\Lambda} = V_0 + \sum_{i=1}^{n} \chi_i. \tag{4.1.39}$$

Let n_j be the multiplicity of χ_{i_j}. By the algebraic decomposition of $V_{N,T,\Lambda}$, $V_{N_0,T,\Lambda}$ can be rewritten as

$$V_{N_0,T,\Lambda} = V_0 + \bigoplus_{j=1}^{k} \chi_{i_j}[n_j]. \tag{4.1.40}$$

We denote

$$V^{(j)} = V_0 + \chi_{i_j}[n_j], \tag{4.1.41}$$

which is a submodule of $V_{N_0,T,\Lambda}$ and admits the filtration

$$0 \subset V_0^{(j)} \subset V_1^{(j)} \subset V_2^{(j)} \subset \cdots \subset V_{n_j}^{(j)} = V^{(j)}, \tag{4.1.42}$$

where $V_0^{(j)} = V_0$, $V_k^{(j)} = V_0 + \chi_{i_j}[k]$, and $1 \le k \le n_j$.

Lemma 4.1.17. *Let (π, V) be an irreducible, admissible representation of* $\mathrm{GSp}(4, F)$, *the decomposition of $V_{N_0,T,\Lambda}$ be as in* (4.1.40). *Suppose that $V_{N_0,T,\Lambda}$ does not contain a one-dimensional submodule W on which H acts via χ_{i_j}. Then $\chi_{i_j}[n_j] \subset \mathcal{K}(\pi)/\mathcal{S}(F^\times)$.*

Proof. First, we will prove the lemma for the case $n_j = 1$. We denote $\chi = \chi_{i_j}$. Let $v \in V$ be such that

$$\pi(\begin{bmatrix} x & & & \\ & x & & \\ & & 1 & \\ & & & 1 \end{bmatrix})v + V(N,T,\Lambda) = \chi(x)v + V(N,T,\Lambda).$$

Assume that $\mathbb{C} \cdot \chi \not\subset \mathcal{K}(\pi)/\mathcal{S}(F^\times)$. Via the homomorphism (4.1.22), $\varphi_v = 0$ in $\mathcal{K}(\pi)/\mathcal{S}(F^\times)$. By Lemma 4.1.10, there exists an $v_0 \in V(N,T,\Lambda)$ such that $\varphi_v = \varphi_{v_0}$, i.e, $\varphi_{v-v_0} = 0$. Hence,

$$\pi(\begin{bmatrix} x & & & \\ & x & & \\ & & 1 & \\ & & & 1 \end{bmatrix})(v - v_0) \in V(N,T,\Lambda \otimes \psi), \text{ for every } x \in F^\times.$$

It follows

$$v - v_0 \in V(N, T, \Lambda \otimes \psi^x), \text{ for every } x \in F^\times, \tag{4.1.43}$$

where $\psi^x(a) = \psi(xa)$, for every $a \in F$. We denote $w = v - v_0$. Let $\bar{w}$, $\bar{v}$, $\bar{v}_0$ be the images of w,v, v_0 in $V_{N_0,T,\Lambda}$. For any $x \in F^\times$, there exist $\bar{v}_x \in V_0$ such that

$$\pi(\begin{bmatrix} x &&& \\ & x && \\ && 1 & \\ &&& 1 \end{bmatrix})\bar{v} = \chi(x)\bar{v} + \bar{v}_x.$$

On the other hand,

$$\begin{aligned} \pi(\begin{bmatrix} x &&& \\ & x && \\ && 1 & \\ &&& 1 \end{bmatrix})\bar{w} &= \chi(x)\bar{v} + \bar{v}_x - \pi(\begin{bmatrix} x &&& \\ & x && \\ && 1 & \\ &&& 1 \end{bmatrix})\bar{v}_0 \\ &= \chi(x)(\bar{v} - \bar{v}_0) + \bar{v}_x - (\pi(\begin{bmatrix} x &&& \\ & x && \\ && 1 & \\ &&& 1 \end{bmatrix})\bar{v}_0 - \chi(x)\bar{v}_0) \\ &= \chi(x)\bar{w} + \bar{v}_x - (\pi(\begin{bmatrix} x &&& \\ & x && \\ && 1 & \\ &&& 1 \end{bmatrix})\bar{v}_0 - \chi(x)\bar{v}_0). \end{aligned}$$

By (4.1.43),

$$\bar{w} \in \bigcap_{x \in F^\times} V_{N_0,T,\Lambda}(N', \psi^x), \tag{4.1.44}$$

which is invariant under the action of H. Hence,

$$\bar{v}_x - (\pi(\begin{bmatrix} x &&& \\ & x && \\ && 1 & \\ &&& 1 \end{bmatrix})\bar{v}_0 - \chi(x)\bar{v}_0) \in \bigcap_{x \in F^\times} V_{N_0,T,\Lambda}(N', \psi^x).$$

But $v_0 \in V(N, T, \Lambda)$, $\bar{v}_0 \in V_0 = V_{N_0,T,\Lambda}(N')$. Hence,

$$\bar{v}_x - (\pi(\begin{bmatrix} x &&& \\ & x && \\ && 1 & \\ &&& 1 \end{bmatrix})\bar{v}_0 - \chi(x)\bar{v}_0) \in \bigcap_{x \in F} V_{N_0,T,\Lambda}(N', \psi^x).$$

By Corollary 2 on page 58 of [4], $\bar{v}_x - (\pi(\begin{bmatrix} x&&&\\ &x&&\\ &&1&\\ &&&1\end{bmatrix})\bar{v}_0 - \chi(x)\bar{v}_0) = 0$, i.e,

$$\pi(\begin{bmatrix} x&&&\\ &x&&\\ &&1&\\ &&&1\end{bmatrix})\bar{w} = \chi(x)\bar{w}.$$

Since $\bar{w} \notin V_0$, $V_0 + \mathbb{C}\bar{w} = V_0 \oplus \mathbb{C}\bar{w} = V_0 \oplus \mathbb{C}\cdot\chi \subset V_{N_0,T,\Lambda}$ as H-modules, which is a contradiction.

For the case $n_j = 2$, we denote $\chi = \chi_{i_j}$. It is similar to prove that $\mathbb{C}\cdot\chi \subset \mathcal{K}(\pi)/\mathcal{S}(F^\times)$. Assume that $\chi[2] \not\subset \mathcal{K}(\pi)/\mathcal{S}(F^\times)$. Now we consider the submodule $V_2^{(j)} = V_0 + \chi[2]$ of $V_{N_0,T,\Lambda}$ as in (4.1.42). By the algebraic structure of $V_{N,T,\Lambda}$, there exist v_1, and v_2 such that

$$\begin{aligned} &\pi(\begin{bmatrix} x&&&\\ &x&&\\ &&1&\\ &&&1\end{bmatrix})v_1 + V(N,T,\Lambda) = \chi(x)v_1 + V(N,T,\Lambda),\\ &\pi(\begin{bmatrix} x&&&\\ &x&&\\ &&1&\\ &&&1\end{bmatrix})v_2 + V(N,T,\Lambda) = \chi(x)v_2 + \chi(x)v(x)v_1 + V(N,T,\Lambda), \end{aligned}$$

$\varphi_{v_1} \notin \mathcal{S}(F^\times)$, and $\varphi_{v_2} \in \mathcal{S}(F^\times)$. We denote $w_2 = \pi(\begin{bmatrix} x&&&\\ &x&&\\ &&1&\\ &&&1\end{bmatrix})v_2$. Hence, there exists $v_0 \in V(N,T,\Lambda)$ such that $w_2 = \chi(x)v_2 + \chi(x)v(x)v_1 + v_0$. It follows

$$\varphi_{w_2} = \chi(x)\varphi_{v_2} + \chi(x)v(x)\varphi_{v_1} + \varphi_{v_0}$$

Since $\varphi_{v_2} \in \mathcal{S}(F^\times)$, $\varphi_{w_2} \in \mathcal{S}(F^\times)$. Also, by Lemma 4.1.10, $\varphi_{v_0} \in \mathcal{S}(F^\times)$. Hence, if we choose x such that $v(x) \neq 0$, $\varphi_{v_1} \in \mathcal{S}(F^\times)$, which is a contradiction. It follows $\chi[2] \subset \mathcal{K}(\pi)/\mathcal{S}(F^\times)$.

Similarly, by induction, one can prove $\chi_{i_j}[n_j] \subset \mathcal{K}(\pi)/\mathcal{S}(F^\times)$. $\square$

Let (ρ, W) be a smooth representation of $\mathrm{GL}(2,F)$, σ be a character of $F^\times$ and $I = \rho \rtimes \sigma$ be a Siegel parabolic induced representation of $\mathrm{GSp}(4,F)$. By Section

5.2 of [11], it is straightforward to obtain the filtration of HTN_0N'-modules of I by restricting the action of P to HTN_0N'

$$0 \subset I_3 \subset I_2 \subset I_1 \subset I_0 = I, \tag{4.1.45}$$

where

- $I_3 \cong \mathcal{S}(F^3, W)$ the space of Schwartz-Bruhat functions which admits the following actions of N, T and H

$$\begin{aligned}
(\begin{bmatrix} 1 & & a & b \\ & 1 & b & c \\ & & 1 & \\ & & & 1 \end{bmatrix} f)(x,y,z) &= f(x+a, y+b, z+c), \\
(\begin{bmatrix} a & & & \\ & b & & \\ & & b & \\ & & & a \end{bmatrix} f)(x,y,z) &= \rho(\begin{bmatrix} b & \\ & a \end{bmatrix})\sigma(ab) f(xb/a, y, za/b), \\
(\begin{bmatrix} a & & & \\ & a & & \\ & & 1 & \\ & & & 1 \end{bmatrix} f)(x,y,z) &= |a|^{-3/2}\sigma(a) f(a^{-1}x, a^{-1}y, a^{-1}z),
\end{aligned}$$

  for every $f \in \mathcal{S}(F^3, W)$.

- $I_2/I_3 \cong \mathcal{S}(F^\times \times F, W)$ the space of compactly supported functions with values in W. It admits the following actions of N, T and H

$$\begin{aligned}
(\begin{bmatrix} 1 & & a & b \\ & 1 & b & c \\ & & 1 & \\ & & & 1 \end{bmatrix} f)(x,y) &= \rho(\begin{bmatrix} 1 & b+xa \\ & 1 \end{bmatrix}) f(x, y + ax^2 + 2bx + c), \\
(\begin{bmatrix} a & & & \\ & b & & \\ & & b & \\ & & & a \end{bmatrix} f)(x,y) &= \sigma(ab)\omega_\rho(a)|a/b|^{3/2} f(ax/b, ay/b), \\
(\begin{bmatrix} a & & & \\ & a & & \\ & & 1 & \\ & & & 1 \end{bmatrix} f)(x,y) &= \sigma(a)\rho(\begin{bmatrix} a & \\ & 1 \end{bmatrix}) f(x, a^{-1}y),
\end{aligned}$$

  where $f \in \mathcal{S}(F^\times \times F, W)$ and ω_ρ is the central character of ρ.

- I_1/I_2 admits the filtration

$$0 \subset I_{12} \subset I_1/I_2$$

such that $I_{12} \cong \mathcal{S}(F, W)$ and $(I_1/I_2)/I_{12} \cong \mathcal{S}(F, W)$, where $\mathcal{S}(F, W)$ is the space of compactly supported functions. We have I_{12} admits the following actions of N, T and H

$$\begin{aligned}
(\begin{bmatrix} 1 & & a & b \\ & 1 & b & c \\ & & 1 & \\ & & & 1 \end{bmatrix} f)(x) &= \rho(\begin{bmatrix} 1 & b \\ & 1 \end{bmatrix}) f(x+c), \\
(\begin{bmatrix} a & & & \\ & b & & \\ & & b & \\ & & & a \end{bmatrix} f)(x) &= \sigma(ab)\omega_\rho(a)|a/b|^{3/2} f(ax/b), \\
(\begin{bmatrix} a & & & \\ & a & & \\ & & 1 & \\ & & & 1 \end{bmatrix} f)(x) &= \sigma(a)\rho(\begin{bmatrix} a & \\ & 1 \end{bmatrix}) f(a^{-1}x),
\end{aligned}$$

for every $f \in \mathcal{S}(F, W)$. Besides, $(I_1/I_2)/I_{12}$ admits the following actions of N, T and H

$$\begin{aligned}
(\begin{bmatrix} 1 & & a & b \\ & 1 & b & c \\ & & 1 & \\ & & & 1 \end{bmatrix} g)(x) &= \rho(\begin{bmatrix} 1 & b \\ & 1 \end{bmatrix}) g(x+a), \\
(\begin{bmatrix} a & & & \\ & b & & \\ & & b & \\ & & & a \end{bmatrix} g)(x) &= \sigma(ab)\omega_\rho(b)|b/a|^{3/2} g(bx/a), \\
(\begin{bmatrix} a & & & \\ & a & & \\ & & 1 & \\ & & & 1 \end{bmatrix} g)(x) &= \sigma(a)\rho(\begin{bmatrix} a & \\ & 1 \end{bmatrix}) g(a^{-1}x),
\end{aligned}$$

for every $g \in \mathcal{S}(F, W)$.

- $I_0/I_1 \cong W$ which admits the trivial action of N and the following actions of T and H

$$\begin{bmatrix} a & & & \\ & b & & \\ & & b & \\ & & & a \end{bmatrix} w = \sigma(ab)\rho(\begin{bmatrix} a & \\ & b \end{bmatrix})w,$$

$$\begin{bmatrix} a & & & \\ & a & & \\ & & 1 & \\ & & & 1 \end{bmatrix} w = |a|^{3/2}\sigma(a)\omega_\rho(a)w,$$

for every $w \in W$.

Since N_0 is exhausted by its compact subgroups, $(\cdot)_{N_0}$ is a compact functor. Hence, by the filtration (4.1.45) of I, we obtain the filtration of I_{N_0}

$$0 \subset (I_3)_{N_0} \subset (I_2)_{N_0} \subset (I_1)_{N_0} \subset (I_0)_{N_0} = I_{N_0}, \tag{4.1.46}$$

where

- $(I_3)_{N_0} \cong \mathcal{S}(F, W)$ which admits the following actions of N', T and H

$$\begin{aligned}
(\begin{bmatrix} 1 & & & b \\ & 1 & b & \\ & & 1 & \\ & & & 1 \end{bmatrix} f)(y) &= f(y+b), \\
(\begin{bmatrix} a & & & \\ & b & & \\ & & b & \\ & & & a \end{bmatrix} f)(y) &= \rho(\begin{bmatrix} b & \\ & a \end{bmatrix})\sigma(ab)f(y), \\
(\begin{bmatrix} a & & & \\ & a & & \\ & & 1 & \\ & & & 1 \end{bmatrix} f)(y) &= |a|^{1/2}\sigma(a)f(a^{-1}y),
\end{aligned}$$

  for every $f \in \mathcal{S}(F, W)$.

- $(I_2)_{N_0}/(I_3)_{N_0} \cong (I_2/I_3)_{N_0} \cong \mathcal{S}(F^\times, J(W))$, where $J(W)$ is the Jacquet module of W with respect to the unipotent radical of the Borel subgroup of $\mathrm{GL}(2, F)$. $\mathcal{S}(F^\times, J(W))$ admits actions of N', T and H as follows

$$\begin{aligned}
(\begin{bmatrix} 1 & & & b \\ & 1 & b & \\ & & 1 & \\ & & & 1 \end{bmatrix} f)(x) &= f(x), \\
(\begin{bmatrix} a & & & \\ & b & & \\ & & b & \\ & & & a \end{bmatrix} f)(x) &= \sigma(ab)\omega_\rho(a)|a/b|^{1/2}J(\rho)(\begin{bmatrix} a/b & \\ & 1 \end{bmatrix})f(ax/b), \\
(\begin{bmatrix} a & & & \\ & a & & \\ & & 1 & \\ & & & 1 \end{bmatrix} f)(x) &= |a|\sigma(a)J(\rho)(\begin{bmatrix} a & \\ & 1 \end{bmatrix})f(x),
\end{aligned}$$

for every $f \in \mathcal{S}(F^\times, J(W))$.

- $(I_1)_{N_0}/(I_2)_{N_0} \cong (I_1/I_2)_{N_0}$ which admits the filtration

$$0 \subset (I_{12})_{N_0} \subset (I_1/I_2)_{N_0},$$

where $(I_{12})_{N_0} \cong W$ admits the following actions of N', T and H

$$\begin{bmatrix} 1 & & & b \\ & 1 & b & \\ & & 1 & \\ & & & 1 \end{bmatrix} w_1 = \rho(\begin{bmatrix} 1 & b \\ & 1 \end{bmatrix})w_1,$$
$$\begin{bmatrix} a & & & \\ & b & & \\ & & b & \\ & & & a \end{bmatrix} w_1 = \sigma(ab)\omega_\rho(a)|a/b|^{1/2}w_1,$$
$$\begin{bmatrix} a & & & \\ & a & & \\ & & 1 & \\ & & & 1 \end{bmatrix} w_1 = |a|\sigma(a)\rho(\begin{bmatrix} a & \\ & 1 \end{bmatrix})w_1,$$

for every $w_1 \in W$. Besides, $(I_1/I_2)_{N_0}/(I_{12})_{N_0} \cong W$ admits the following actions of N', T and H

$$\begin{bmatrix} 1 & & & b \\ & 1 & b & \\ & & 1 & \\ & & & 1 \end{bmatrix} w_2 = \rho(\begin{bmatrix} 1 & b \\ & 1 \end{bmatrix})w_2,$$
$$\begin{bmatrix} a & & & \\ & b & & \\ & & b & \\ & & & a \end{bmatrix} w_2 = \sigma(ab)\omega_\rho(b)|b/a|^{1/2}w_2,$$
$$\begin{bmatrix} a & & & \\ & a & & \\ & & 1 & \\ & & & 1 \end{bmatrix} w_2 = |a|\sigma(a)\rho(\begin{bmatrix} a & \\ & 1 \end{bmatrix})w_2,$$

for every $w_2 \in W$.

- $(I_0)_{N_0}/(I_1)_{N_0} \cong (I_0/I_1)_{N_0} \cong I_0/I_1 \cong W$ which admits the trivial action of N' and the following actions of T and H

$$\begin{bmatrix} a & & & \\ & b & & \\ & & b & \\ & & & a \end{bmatrix} w = \sigma(ab)\rho(\begin{bmatrix} a & \\ & b \end{bmatrix})w,$$

$$\begin{bmatrix} a & & & \\ & a & & \\ & & 1 & \\ & & & 1 \end{bmatrix} w = |a|^{3/2}\sigma(a)\omega_\rho(a)w,$$

for every $w \in W$.

Let ρ be infinitely dimensional and $\Lambda = (\Lambda_1, \Lambda_2)$ be a character of T. Then

- $(I_3)_{N_0,T,\Lambda} \cong ((I_3)_{N_0})_{T,\Lambda} \cong \mathcal{S}(F)$ which admits the following actions of H and N'

$$(\begin{bmatrix} 1 & & & b \\ & 1 & b & \\ & & 1 & \\ & & & 1 \end{bmatrix} f)(y) = f(y+b),$$
$$(\begin{bmatrix} a & & & \\ & a & & \\ & & 1 & \\ & & & 1 \end{bmatrix} f)(y) = |a|^{1/2}\sigma(a)f(a^{-1}y),$$

for every $f \in \mathcal{S}(F)$.

- $((I_2)_{N_0}/(I_3)_{N_0})_{T,\Lambda} \cong \nu\chi \otimes J(\rho)$ as H-modules and admit the trivial action of N'. More explicitly,

$$\begin{bmatrix} a & & & \\ & a & & \\ & & 1 & \\ & & & 1 \end{bmatrix} u = |a|\sigma(a)J(\rho)(\begin{bmatrix} a & \\ & 1 \end{bmatrix})u,$$

for every $u \in ((I_2)_{N_0}/(I_3)_{N_0})_{T,\Lambda}$.

- $((I_1)_{N_0}/(I_2)_{N_0})_{T,\Lambda}$ depends on the Λ. There are three possibilities:
  - Case 1. If $\Lambda \neq (\nu^{1/2}\sigma\omega_\rho, \nu^{-1/2}\sigma)$ and $\Lambda \neq (\nu^{-1/2}\sigma, \nu^{1/2}\sigma\omega_\rho)$, then

$$((I_1)_{N_0}/(I_2)_{N_0})_{T,\Lambda} = 0.$$

  - Case 2. If "$\Lambda = (\nu^{1/2}\sigma\omega_\rho, \nu^{-1/2}\sigma)$, $\Lambda \neq (\nu^{-1/2}\sigma, \nu^{1/2}\sigma\omega_\rho)$" or "$\Lambda \neq (\nu^{1/2}\sigma\omega_\rho, \nu^{-1/2}\sigma)$, $\Lambda = (\nu^{-1/2}\sigma, \nu^{1/2}\sigma\omega_\rho)$", $((I_1)_{N_0}/(I_2)_{N_0})_{T,\Lambda} = W$ which admits the following actions of H and N'

$$\begin{bmatrix} 1 & & & b \\ & 1 & b & \\ & & 1 & \\ & & & 1 \end{bmatrix} w = \rho(\begin{bmatrix} 1 & b \\ & 1 \end{bmatrix})w,$$
$$\begin{bmatrix} a & & & \\ & a & & \\ & & 1 & \\ & & & 1 \end{bmatrix} w = |a|\sigma(a)\rho(\begin{bmatrix} a & \\ & 1 \end{bmatrix})w,$$

for every $w \in W$.

– Case 3. If $\Lambda = (\nu^{1/2}\sigma\omega_\rho, \nu^{-1/2}\sigma) = (\nu^{-1/2}\sigma, \nu^{1/2}\sigma\omega_\rho)$, then

$$((I_1)_{N_0}/(I_2)_{N_0})_{T,\Lambda} = W + W,$$

where W admits the same actions of H and N' as in the previous case.

- $((I_0)_{N_0}/(I_1)_{N_0})_{T,\Lambda} \cong W_{T,\Lambda}$ the Waldspurger module of W which is one dimensional and admits the trivial action of N' and the following action of H

$$\begin{bmatrix} a & & & \\ & a & & \\ & & 1 & \\ & & & 1 \end{bmatrix} w = |a|^{3/2}\sigma(a)\omega_\rho(a)w,$$

for every $w \in W$.

Let us consider $\mathcal{S}(F)$ the space of Schwartz-Bruhat functions on F with respect to representations τ_1^χ, τ_2^χ of HN' as follows:

$$(\tau_1^\chi(\begin{bmatrix} a & & & \\ & a & & \\ & & 1 & \\ & & & 1 \end{bmatrix})f)(x) = \chi(a)f(ax), \text{ and } (\tau_1^\chi(\begin{bmatrix} 1 & & & b \\ & 1 & b & \\ & & 1 & \\ & & & 1 \end{bmatrix})f)(x) = \psi(bx)f(x),$$
$$(\tau_2^\chi(\begin{bmatrix} a & & & \\ & a & & \\ & & 1 & \\ & & & 1 \end{bmatrix})f)(x) = |a|^{-1}\chi(a)f(a^{-1}x), \text{ and } (\tau_2^\chi(\begin{bmatrix} 1 & & & b \\ & 1 & b & \\ & & 1 & \\ & & & 1 \end{bmatrix})f)(x) = f(x+b),$$

for every $f \in \mathcal{S}(F^\times)$. We have the following proposition.

Proposition 4.1.18. *Let V, V_0, $(\tau_1^\chi, \mathcal{S}(F))$ and $(\tau_2^\chi, \mathcal{S}(F))$ as above. Then*

$$V_0 \hookrightarrow \tau_1^\chi \cong \tau_2^\chi. \tag{4.1.47}$$

Furthermore, there is a filtration $0 \subset W \subset \tau_2^\chi$, where $W \cong V_0$ and $\tau_2^\chi / W \cong \chi$, and there is no one-dimensional subspace of τ_2^χ on which H acts via χ.

Proof. By (4.1.36), $V_{N_0,T,\Lambda}(N') \cong \text{c-Ind}_{N'}^{HN'} \psi$ which is isomorphic to $\mathcal{S}(F^\times)$ with the actions:

$$(\begin{bmatrix} a &&& \\ & a && \\ && 1 & \\ &&& 1 \end{bmatrix} f)(x) = f(ax), \text{ and } (\begin{bmatrix} 1 &&& b \\ & 1 & b & \\ && 1 & \\ &&& 1 \end{bmatrix} f)(x) = \psi(bx) f(x).$$

Hence, $V_0 \cong \mathcal{S}(F^\times) \hookrightarrow \tau_1^\chi$ by $f \mapsto \chi^{-1} f$. On the other hand, $\tau_1^\chi \cong \tau_2^\chi$ by the following isomorphism

$$\begin{aligned} \tau_2^\chi \to & \qquad \tau_1^\chi \\ f \mapsto & (x \mapsto \int_F f(y) \psi^{-1}(xy) dy). \end{aligned}$$

The last statement is easy to see from the definition of τ_2^χ. $\square$

Let (π, V) be a generic representation of $\mathrm{GSp}(4, F)$. A character χ is called *special* with respect to π if χ is described as in the Table 4.3.

Lemma 4.1.19. *Let (π, V) be a generic representation of $\mathrm{GSp}(4, F)$, $\Lambda = (\Lambda_1, \Lambda_2)$ be such that Λ_1 is not special as described in the Table 4.3, and $V_{N_0,T,\Lambda} = V_0 + \bigoplus_{j=1}^k \chi_{i_j}[n_j]$ be the decompostion of $V_{N_0,T,\Lambda}$ as described in (4.1.40). Then $V_{N_0,T,\Lambda}$ does not contain a one-dimensional submodule W on which H acts via χ_{i_j}, for $1 \le j \le k$.*

Table 4.3: Special characters of generic representations of GSp(4). An entry "—" indicates that no special character exists.

		representation	special characters
I		$\chi_1 \times \chi_2 \rtimes \sigma$	$\nu^{1/2}\sigma\chi_1\chi_2, \nu^{-1/2}\sigma\chi_1\chi_2, \nu^{1/2}\sigma\chi_1, \nu^{-1/2}\sigma\chi_1,$ $\nu^{1/2}\sigma\chi_2, \nu^{-1/2}\sigma\chi_2, \nu^{1/2}\sigma, \nu^{-1/2}\sigma$
II	a	$\chi\mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$	$\nu^{1/2}\chi^2\sigma, \nu^{-1/2}\chi^2\sigma, \nu^{1/2}\sigma,$ $\nu^{-1/2}\sigma, \chi\sigma$
III	a	$\chi \rtimes \sigma\mathrm{St}_{\mathrm{GSp}(2)}$	$\chi\sigma, \sigma$
IV	a	$\sigma\mathrm{St}_{\mathrm{GSp}(4)}$	$\nu^{-1}\sigma, \nu\sigma$
V	a	$\delta([\xi, \nu\xi], \nu^{-1/2}\sigma)$	$\xi\sigma, \sigma$
VI	a	$\tau(S, \nu^{-1/2}\sigma)$	σ
VII		$\chi \rtimes \pi$	—
VIII	a	$\tau(S, \pi)$	—
IX	a	$\delta(\nu\xi, \nu^{-1/2}\pi(\mu))$	—
X		$\delta(\nu\xi, \nu^{-1/2}\pi(\mu))$	$\nu^{1/2}\omega_\pi\sigma, \nu^{-1/2}\sigma, \nu^{-1/2}\omega_\pi\sigma, \nu^{1/2}\sigma$
XI	a	$\delta(\nu\xi, \nu^{-1/2}\pi(\mu))$	σ
		generic supercuspidal	—

Proof. We will prove the lemma case by case. The key fact is that we compare $((I_{i-1})_{N_0}/(I_i)_{N_0})_{T,\Lambda}$ with the filtration (4.1.38) to explicitly describe $V_{N_0T,\Lambda}$. Since we consider Λ_1 as a non-special character, $((I_1)_{N_0}/(I_2)_{N_0})_{T,\Lambda} = (I_1/I_2)_{N_0,T,\Lambda} = 0$.

- Type I. $\rho = \chi_1 \times \chi_2$, $\pi = \rho \rtimes \sigma$ and $I = V$. Since Λ is not special, $\Lambda \neq (\nu^{1/2}\sigma\chi_1\chi_2, \nu^{-1/2}\sigma)$ and $\Lambda \neq (\nu^{-1/2}\sigma, \nu^{1/2}\sigma\chi_1\chi_2)$, $((I_1)_{N_0}/(I_2)_{N_0})_{T,\Lambda} = 0$. We have

$$0 \subset (I_3)_{N_0} \subset (I_2)_{N_0} \subset (I_1)_{N_0} \subset (I_0)_{N_0} = V_{N_0}.$$

Let us consider the short exact sequence

$$0 \to (I_3)_{N_0} \to (I_2)_{N_0} \to (I_2/I_3)_{N_0} \to 0.$$

Applying the functor $(-)_{T,\Lambda}$, we get

$$(I_3)_{N_0,T,\Lambda} \xrightarrow{\alpha} (I_2)_{N_0,T,\Lambda} \to (I_2/I_3)_{N_0,T,\Lambda} \to 0$$

We claim that α is injective. In fact, $(I_3)_{N_0,T,\Lambda} \cong \tau_2^{\nu^{3/2}\sigma}$. By Proposition 4.1.18, $(I_3)_{N_0,T,\Lambda}$ contains only V_0 as a HN'-module. Hence, if $\ker(\alpha) \neq 0$, i.e, $V_0 \subset \ker(\alpha)$, then $(I_2)_{N_0,T,\Lambda}$ is finite-dimensional. It follows $V_{N_0,T,\Lambda}$ is finite-dimensional, which is a contradiction. Hence, α is injective, which implies

$$(I_2)_{N_0,T,\Lambda}/(I_3)_{N_0,T,\Lambda} \cong (I_2/I_3)_{N_0,T,\Lambda},$$

and $(I_2)_{N_0,T,\Lambda} = \tau_2^{\nu^{3/2}\sigma} + \nu^{3/2}\sigma\chi_1 + \nu^{3/2}\sigma\chi_2$. Next, we want to show that β is injective, where β is defined in the following short exact sequence

$$(I_2)_{N_0,T,\Lambda} \xrightarrow{\beta} (I_1)_{N_0,T,\Lambda} \to (I_1/I_2)_{N_0,T,\Lambda} \to 0.$$

Assume that $\ker(\beta) \neq 0$. Similarly, one can prove that $\tau_2^{\nu^{3/2}\sigma} \not\subset \ker(\beta)$. For now, we do not know whether $\nu^{3/2}\sigma\chi_1$, $\nu^{3/2}\sigma\chi_2$, or $\nu^{3/2}\sigma\chi_1 + \nu^{3/2}\sigma\chi_2$ is a HN'-submodule of $(I_2)_{N_0,T,\Lambda}$. In fact, they are not HN'-submodules of $(I_2)_{N_0,T,\Lambda}$, which will be proven. Assume that $\ker(\beta)$ contains either $\nu^{3/2}\sigma\chi_1$, $\nu^{3/2}\sigma\chi_2$, or $\nu^{3/2}\sigma\chi_1+\nu^{3/2}\sigma\chi_2$ as a HN'-submodule. It follows $(I_1)_{N_0,T,\Lambda}$ does not contain $\nu^{3/2}\sigma\chi_1$, $\nu^{3/2}\sigma\chi_2$, or $\nu^{3/2}\sigma\chi_1+\nu^{3/2}\sigma\chi_2$. Let us consider the short exact sequence

$$(I_1)_{N_0,T,\Lambda} \xrightarrow{\gamma} (I_0)_{N_0,T,\Lambda} = V_{N_0,T,\Lambda} \to (I_0/I_1)_{N_0,T,\Lambda} \to 0.$$

Hence, $V_{N_0,T,\Lambda}$ does not contain $\nu^{3/2}\sigma\chi_1$, $\nu^{3/2}\sigma\chi_2$, or $\nu^{3/2}\sigma\chi_1 + \nu^{3/2}\sigma\chi_2$, which is a contradiction. Hence, β is injective, which implies

$$(I_1)_{N_0,T,\Lambda}/(I_2)_{N_0,T,\Lambda} \cong (I_1/I_2)_{N_0,T,\Lambda} = 0,$$

and $(I_1)_{N_0,T,\Lambda} = (I_2)_{N_0,T,\Lambda} = \tau_2^{\nu^{3/2}\sigma} + \nu^{3/2}\sigma\chi_1 + \nu^{3/2}\sigma\chi_2$. Similarly, one can prove

$$(I_0)_{N_0,T,\Lambda}/(I_1)_{N_0,T,\Lambda} \cong (I_0/I_1)_{N_0,T,\Lambda} \cong \nu^{3/2}\sigma\chi_1\chi_2,$$

and $V_{N_0,T,\Lambda} = (I_0)_{N_0,T,\Lambda} = \tau_2^{\nu^{3/2}\sigma} + \nu^{3/2}\sigma\chi_1 + \nu^{3/2}\sigma\chi_2 + \nu^{3/2}\sigma\chi_1\chi_2$. More explicitly,

$$0 \subset (I_3)_{N_0,T,\Lambda} \subset (I_2)_{N_0,T,\Lambda} = (I_1)_{N_0,T,\Lambda} \subset (I_0)_{N_0,T,\Lambda} = V_{N_0,T,\Lambda},$$

where $(I_3)_{N_0,T,\Lambda} \cong \tau_2^{\nu^{3/2}}$, $(I_2)_{N_0,T,\Lambda}/(I_3)_{N_0,T,\Lambda} \cong \nu^{3/2}\sigma\chi_1 + \nu^{3/2}\sigma\chi_2$, and $(I_0)_{N_0,T,\Lambda}/(I_1)_{N_0,T,\Lambda} \cong \nu^{3/2}\sigma\chi_1\chi_2$.

Assume that $V_{N_0,T,\Lambda}$ contains a one-dimensional submodule $W = \langle w \rangle$ on which H acts via $\nu^{3/2}\sigma$. We consider the projection $\bar{w}$ of w on $(I_0)_{N_0,T,\Lambda}/(I_1)_{N_0,T,\Lambda}$. By the algebraic structure of $V_{N,T,\Lambda}$, without loss of generality, we may assume χ_1, χ_2, $\chi_1\chi_2$, σ and 1 are pairwise different. We choose $x \in F^\times$ such that $(\chi_1\chi_2)(x) \neq 1$. It follows

$$(\nu^{3/2}\sigma)(x)\bar{w} = \overline{\pi(\begin{bmatrix} x &&& \\ & x && \\ && 1 & \\ &&& 1 \end{bmatrix})w} = (\nu^{3/2}\sigma\chi_1\chi_2)(x)\bar{w},$$

i.e, $\bar{w}$ is zero on $(I_0)_{N_0,T,\Lambda}/(I_1)_{N_0,T,\Lambda}$. Similarly, the projection of w is zero on $(I_2)_{N_0,T,\Lambda}/(I_3)_{N_0,T,\Lambda}$. Hence, $W \subset (I_3)_{N_0,T,\Lambda} \cong \tau_2^{\nu^{3/2}\sigma}$, which is a contradiction by Proposition 4.1.18. Hence, $V_{N_0,T,\Lambda}$ does not contain a one-dimensional submodule on which H acts via $\nu^{3/2}\sigma$

On the other hand, by the isomorphism $\pi \cong \chi_1^{-1} \times \chi_2 \rtimes \chi_1\sigma$ with the conditions $\Lambda_1 \neq \nu^{1/2}\chi_2\sigma$ and $\Lambda_1 \neq \nu^{-1/2}\chi_1\sigma$, it is similar to show that $V_{N_0,T,\Lambda} = \tau_2^{\nu^{3/2}\sigma\chi_1} + \nu^{3/2}\sigma + \nu^{3/2}\sigma\chi_2 + \nu^{3/2}\sigma\chi_1\chi_2$. Hence, $V_{N_0,T,\Lambda}$ does not contain a one-dimensional submodule W on which H acts via $\nu^{3/2}\sigma\chi_1$.

Similarly, we make use of the isomorphisms

$$\pi \cong \chi_1 \times \chi_2^{-1} \rtimes \chi_2\sigma \cong \chi_1^{-1} \times \chi_2^{-1} \rtimes \chi_1\chi_2\sigma.$$

and the conditions $\Lambda_1 \neq \nu^{1/2}\chi_1\sigma$, $\Lambda_1 \neq \nu^{-1/2}\chi_2\sigma$, $\Lambda_1 \neq \nu^{1/2}\sigma$ and $\Lambda_1 \neq \nu^{-1/2}\chi_1\chi_2\sigma$ to prove that $V_{N_0,T,\Lambda}$ does not contain one-dimensional submodules W_1, and W_2 on which H acts via $\nu^{3/2}\sigma\chi_2$, and $\nu^{3/2}\sigma\chi_1\chi_2$, respectively.

- Type IIa. $\rho = \chi\mathrm{St}_{\mathrm{GL}(2)}$ and $\pi = \rho \rtimes \sigma$. It is similar to Type I to show that $V_{N_0,T,\Lambda}$ does not contain a one-dimensional submodule W on which H acts via $\nu^{3/2}\sigma$.. By the isomorphism

$$\pi = \chi \mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma \cong \chi^{-1}\mathrm{St}_{\mathrm{GL}(2)} \rtimes \chi^2\sigma, \tag{4.1.48}$$

$V_{N_0,T,\Lambda}$ does not contain a one-dimensional submodule W on which H acts via $\nu^{3/2}\chi^2\sigma$.. On the other hand, , $\pi = \chi\mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma \subset I$.

Now we consider $I = (\chi\nu^{-1/2} \times \chi\nu^{1/2}) \rtimes \sigma \cong (\chi\nu^{-1/2} \times \chi^{-1}\nu^{-1/2}) \rtimes \nu^{1/2}\chi\sigma$ and $\Lambda_1 \neq \chi\sigma$. We know

$$0 \to \chi 1_{\mathrm{GL}(2)} \rtimes \sigma \to I \to V \to 0.$$

It follows $I_{N_0,T,\Lambda} \to V_{N_0,T,\Lambda} \to 0$. Similarly, $V_{N_0,T,\Lambda}$ does not contain a one-dimensional submodule W on which H acts via $\nu^2\chi\sigma$.

- Type IIIa. Let us consider $\rho = \chi \times \nu^{-1}$, $I = \rho \rtimes \nu^{1/2}\sigma$ and $\Lambda_1 \notin \{\chi\sigma, \sigma\}$. Hence,

$$0 \to \chi \rtimes \sigma 1_{\mathrm{GSp}(2)} \to I \to V \to 0.$$

It follows $I_{N_0,T,\Lambda} \to V_{N_0,T,\Lambda} \to 0$. Similarly, $V_{N_0,T,\Lambda}$ does not contain a one-dimensional submodule W on which H acts via $\nu^2\sigma$. On the other hand, $I \cong (\chi^{-1} \times \nu^{-1}) \rtimes \nu^{1/2}\chi\sigma$. It is similar to show $V_{N_0,T,\Lambda}$ does not contain a one-dimensional submodule W on which H acts via $\nu^2\chi\sigma$.

- Type IVa. Let us consider $\rho = \nu^{-3/2}\mathrm{St}_{\mathrm{GL}(2)}$, $I = \rho \rtimes \nu^{3/2}\sigma$, V is a quotient of I and $\Lambda_1 \notin \{\nu^{-1}\sigma, \nu\sigma\}$. Since

$$0 \to L(\nu^{3/2}\mathrm{St}_{GL(2)}, \nu^{-3/2}\sigma) \to I \to V \to 0,$$

it is similar to show that $V_{N_0,T,\Lambda}$ does not contain a one-dimensional submodule W on which H acts via $\nu^3\sigma$.

- Type Va. Let us consider $\rho = \nu^{-1/2}\xi \mathrm{St}_{\mathrm{GL}(2)}$, $I_1 = \rho \rtimes \xi\nu^{1/2}\sigma$, $V = \delta([\xi, \nu\xi], \nu^{-1/2}\sigma)$ and $\Lambda_1 \neq \xi\sigma$. Since

$$0 \to L(\nu^{1/2}\xi \mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\xi\sigma) \to I_1 \to V \to 0,$$

it is similar to show that $V_{N_0,T,\Lambda}$ does not contain a one-dimensional submodule W on which H acts via $\nu^2\xi\sigma$.. On the other hand, V is also an irreducible quotient of $I_2 = \nu^{-1/2}\xi \mathrm{St}_{\mathrm{GL}(2)} \rtimes \nu^{1/2}\sigma$. Since $\Lambda_1 \neq \sigma$, it is similar to prove that $V_{N_0,T,\Lambda}$ does not contain a one-dimensional submodule W on which H acts via $\nu^2\sigma$.

- Type VIa. Let us consider $\rho = \nu^{-1/2}\mathrm{St}_{\mathrm{GL}(2)}$, $I_1 = \rho \rtimes \nu^{1/2}\sigma$, $V = \tau(S, \tau^{-1/2}\sigma)$ and $\Lambda_1 \neq \sigma$. We have the short exact sequence

$$0 \to L(\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma) \to I \to V \to 0,$$

By the algebraic decomposition of $V_{N,T,\Lambda}$ as in Table 3.4, the filtration of $I_{N_0,T,\Lambda}$ with $V_{N_0,T,\Lambda}$, and Lemma 4.1.17, one can similarly show that $V_{N_0,T,\Lambda}$ does not contain a two-dimensional submodule W on which H acts via $(\nu\sigma)[2]$.

- Types VII, VIIIa and IXa: Since the Jacquet modules $V_N = 0$, it is trivial.

- Type X. It is similar to types above to consider $V = I = \pi \rtimes \sigma \cong \tilde{\pi} \rtimes \omega_\pi\sigma$, where π is supercuspidal and $\Lambda_1 \notin \{\nu^{1/2}\omega_\pi\sigma, \nu^{-1/2}\sigma, \nu^{-1/2}\omega_\pi\sigma, \nu^{1/2}\sigma\}$.

- Type XIa. Let us consider $I = \nu^{-1/2}\pi \rtimes \nu^{1/2}\sigma$, where π has the trivial central character and $\Lambda_1 \neq \sigma$. We have the following short exact sequence

$$0 \to L(\nu^{1/2}\pi, \nu^{-1/2}\sigma) \to I \to V \to 0,$$

It is similar to show that$V_{N_0,T,\Lambda}$ does not contain a one-dimensional submodule W on which H acts via $\nu^2\sigma$.

$\square$

Corollary 4.1.20. *Table 4.4 shows the asymptotic behavior of the functions* $B(\mathrm{diag}(u,u,1,1))$ *for all irreducible, admissible representations* (π, V) *of* $\mathrm{GSp}(4,F)$*, where* B *runs through a split* (Λ,β)*-Bessel model of* π *such that* Λ_1 *is not special as in Table 4.3. An entry "—" indicates that no such Bessel model exists.*

Again, for type I we have to distinguish various cases:

$$|u|^{-3/2}\, B(\mathrm{diag}(u,u,1,1)) \tag{4.1.49}$$

$$
= \begin{cases} C_1(\chi_1\chi_2\sigma)(u) + C_2(\chi_1\sigma)(u) + C_3(\chi_2\sigma)(u) + C_4\sigma(u) & \textit{if } \chi_1\chi_2, \chi_1, \chi_2, 1 \textit{ are} \\ & \textit{pairwise different,} \\ C_1(\chi^2\sigma)(u) + (C_2 + C_3 v(u))(\chi\sigma)(u) + C_4\sigma(u) & \textit{if } \chi := \chi_1 = \chi_2 \neq 1, \\ & \chi^2 \neq 1, \\ (C_1 + C_2 v(u))(\chi\sigma)(u) + (C_3 + C_4 v(u))\sigma(u) & \textit{if } \chi := \chi_1 = \chi_2 \neq 1, \\ & \chi^2 = 1, \\ (C_1 + C_2 v(u))(\chi\sigma)(u) + (C_3 + C_4 v(u))\sigma(u) & \textit{if } \{\chi_1, \chi_2\} = \\ & = \{\chi \neq 1, 1\}, \\ (C_1 + C_2 v(u) + C_3 v^2(u) + C_4 v^3(u))\sigma(u) & \textit{if } \chi_1 = \chi_2 = 1. \end{cases}
$$

Proof. It is a consequence of Lemma 4.1.17 and Lemma 4.1.19. □

4.2 Local zeta integrals and L-factors

Given an irreducible, admissible, unitary representation π of $\mathrm{GSp}(4, F)$ and a character μ of $F^\times$, a certain type of zeta integral was introduced in Sect. 3 of [9] and used to define an L-factor $L^{\mathrm{PS}}(s, \pi, \mu)$. These zeta integrals depend on a choice of Bessel model for π, and hence the L-factor may also depend on this choice. In many cases though one can prove that $L^{\mathrm{PS}}(s, \pi, \mu)$ is independent of the choice of Bessel data.

In Sect. 4.2.1 we introduce a simplified type of zeta integral and use it to define the *regular part* $L^{\mathrm{PS}}_{\mathrm{reg}}(s, \pi, \mu)$ of the Piatetski-Shapiro L-factor. The simplified zeta

Table 4.4: Asymptotic behavior of $B(\mathrm{diag}(u,u,1,1))$ in the split case. An entry "—" indicates that no split Bessel functional exists.

		representation		$\lvert u\rvert^{-3/2}\, B(\mathrm{diag}(u,u,1,1))$
I		$\chi_1 \times \chi_2 \rtimes \sigma$		see (4.1.49)
II	a	$\chi\mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$	$\chi^2 \neq 1$	$C_1(\nu^{1/2}\chi\sigma)(u) + C_2(\chi^2\sigma)(u) + C_3\sigma(u)$
			$\chi^2 = 1$	$C_1(\nu^{1/2}\chi\sigma)(u) + (C_2 + C_3 v(u))\sigma(u)$
	b	$\chi 1_{\mathrm{GL}(2)} \rtimes \sigma$	$\chi^2 \neq 1$	$C_1(\nu^{-1/2}\chi\sigma)(u) + C_2(\chi^2\sigma)(u) + C_3\sigma(u)$
			$\chi^2 = 1$	$C_1(\nu^{-1/2}\chi\sigma)(u) + (C_2 + C_3 v(u))\sigma(u)$
III	a	$\chi \rtimes \sigma\mathrm{St}_{\mathrm{GSp}(2)}$		$C_1(\nu^{1/2}\chi\sigma)(u) + C_2(\nu^{1/2}\sigma)(u)$
	b	$\chi \rtimes \sigma 1_{\mathrm{GSp}(2)}$		$C_1(\nu^{-1/2}\chi\sigma)(u) + C_2(\nu^{-1/2}\sigma)(u)$
IV	a	$\sigma\mathrm{St}_{\mathrm{GSp}(4)}$		$C(\nu^{3/2}\sigma)(u)$
	b	$L(\nu^2, \nu^{-1}\sigma\mathrm{St}_{\mathrm{GSp}(2)})$		$C_1(\nu^{3/2}\sigma)(u) + C_2(\nu^{-1/2}\sigma)(u)$
	c	$L(\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-3/2}\sigma)$		$C_1(\nu^{-3/2}\sigma)(u) + C_2(\nu^{1/2}\sigma)(u)$
	d	$\sigma 1_{\mathrm{GSp}(4)}$		—
V	a	$\delta([\xi, \nu\xi], \nu^{-1/2}\sigma)$		$C_1(\nu^{1/2}\xi\sigma)(u) + C_2(\nu^{1/2}\sigma)(u)$
	b	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$		$C_1(\nu^{1/2}\xi\sigma)(u) + C_2(\nu^{-1/2}\sigma)(u)$
	c	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\xi\sigma)$		$C_1(\nu^{-1/2}\xi\sigma)(u) + C_2(\nu^{1/2}\sigma)(u)$
	d	$L(\nu\xi, \xi \rtimes \nu^{-1/2}\sigma)$		—
VI	a	$\tau(S, \nu^{-1/2}\sigma)$		$(C_1 + C_2 v(u))(\nu^{1/2}\sigma)(u)$
	b	$\tau(T, \nu^{-1/2}\sigma)$		—
	c	$L(\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)}, \nu^{-1/2}\sigma)$		$C(\nu^{-1/2}\sigma)(u)$
	d	$L(\nu, 1_{F^\times} \rtimes \nu^{-1/2}\sigma)$		$C_1 + C_2 v(u))(\nu^{-1/2}\sigma)(u)$
VII		$\chi \rtimes \pi$		0
VIII	a	$\tau(S, \pi)$		0
	b	$\tau(T, \pi)$		0
IX	a	$\delta(\nu\xi, \nu^{-1/2}\pi(\mu))$		0
	b	$L(\nu\xi, \nu^{-1/2}\pi(\mu))$		—

representation			$\lvert u\rvert^{-3/2}\, B(\mathrm{diag}(u,u,1,1))$
X	$\pi \rtimes \sigma$	$\omega_\pi \neq 1$	$C_1(\omega_\pi\sigma)(u) + C_2\sigma(u)$
		$\omega_\pi = 1$	$(C_1 + C_2 v(u))\sigma(u)$
XI	a $\delta(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$		$C(\nu^{1/2}\sigma)(u)$
	b $L(\nu^{1/2}\pi, \nu^{-1/2}\sigma)$		$C(\nu^{-1/2}\sigma)(u)$
supercuspidal			0

integrals also depend on the choice of a Bessel model for π. Using the asymptotic behavior given in Table 4.2, we explicitly calculate $L^{\mathrm{PS}}_{\mathrm{reg}}(s,\pi,\mu)$ in the non-split case for all representations. It turns out that $L^{\mathrm{PS}}_{\mathrm{reg}}(s,\pi,\mu)$ is independent of the choice of Bessel model, and coincides with the usual degree-4 (spin) Euler factor if π is generic. For non-generic representations, however, the two factors do not agree in general.

We then investigate the Piatetski-Shapiro zeta integrals (4.2.27). Their definition involves a certain subgroup G of $\mathrm{GSp}(4,F)$, to which we dedicate Sect. 4.2.2. The resulting L-factor $L^{\mathrm{PS}}(s,\pi,\mu)$ is either equal to $L^{\mathrm{PS}}_{\mathrm{reg}}(s,\pi,\mu)$, or has an additional factor $L(s+1/2,\Lambda_\mu)$, where $\Lambda_\mu = \Lambda\cdot(\mu\circ N_{L/F})$ depends on the Bessel data. In Sect. 4.2.5 we will identify several cases where $L^{\mathrm{PS}}(s,\pi,\mu) = L^{\mathrm{PS}}_{\mathrm{reg}}(s,\pi,\mu)$.

Overall in this section we closely follow [9]. However, we treat all representations, not only unitary ones. Our notion of exceptional pole is slightly more general than the one given in [9]. Also, we fill in some of the missing proofs of [9].

4.2.1 The simplified zeta integrals

Let π be an irreducible, admissible representation of $\mathrm{GSp}(4,F)$. Let $\mathcal{B}(\pi,\Lambda,\beta)$ be a (Λ,β)-Bessel model for π. Let μ be a character of $F^\times$. For $B\in\mathcal{B}(\pi,\Lambda,\beta)$ and

$s \in \mathbb{C}$, we define the *simplified zeta integrals*

$$\zeta(s, B, \mu) = \int\limits_{F^\times} B(\left[\begin{smallmatrix} x & \\ & 1 \end{smallmatrix}\right])\mu(x)|x|^{s-3/2}\, d^\times x. \tag{4.2.1}$$

The same integrals appear in Proposition 18 of [6]. Using the general form (4.1.18) of the functions $B(\left[\begin{smallmatrix} x & \\ & 1 \end{smallmatrix}\right])$, which holds both in the split and the non-split case, it is easy to see that $\zeta(s, B, \mu)$ converges to an element $\mathbb{C}(q^{-s})$ for real part of s large enough. Let $I(\pi, \mu)$ be the $\mathbb{C}$-vector subspace of $\mathbb{C}(q^{-s})$ spanned by all $\zeta(s, B, \mu)$ as B runs through $\mathcal{B}(\pi, \Lambda, \beta)$.

Proposition 4.2.1. *Let π be an irreducible, admissible representation of* $\mathrm{GSp}(4, F)$ *admitting a* (Λ, β)*-Bessel model with β as in* (2.1.4)*. Then $I(\pi, \mu)$ is a non-zero $\mathbb{C}[q^{-s}, q^s]$ module containing $\mathbb{C}$, and there exists $R(X) \in \mathbb{C}[X]$ such that $R(q^{-s})I(\pi, \mu) \subset \mathbb{C}[q^{-s}, q^s]$, so that $I(\pi, \mu)$ is a fractional ideal of the principal ideal domain $\mathbb{C}[q^{-s}, q^s]$ whose quotient field is $\mathbb{C}(q^{-s})$. The fractional ideal $I(\pi, \mu)$ admits a generator of the form $1/Q(q^{-s})$ with $Q(0) = 1$, where $Q(X) \in \mathbb{C}[X]$.*

Proof. One can argue as in the proof of Proposition 2.6.4 of [10]. One step in the proof is to show that $I(\pi, \mu)$ contains $\mathbb{C}$. This follows from Lemma 4.1.4. $\square$

Using the notation of this proposition, we set

$$L^{\mathrm{PS}}_{\mathrm{reg}}(s, \pi, \mu) := 1/Q(q^{-s}) \tag{4.2.2}$$

and call this the *regular part of the Piatetski-Shapiro L-factor*; see [9]. As the notation indicates, $L^{\mathrm{PS}}_{\mathrm{reg}}(s, \pi, \mu)$ does not depend on the Bessel data β and Λ. This is implied by the following result.

Theorem 4.2.2. *Table 4.5 shows the factors $L^{\rm PS}_{\rm reg}(s,\pi,\mu)$ for all irreducible, admissible representations (π,V) of $\mathrm{GSp}(4,F)$ in the non-split case. An entry "—" indicates that no non-split Bessel functional exists.*

Proof. Up to an element of $\mathcal{S}(F^\times)$, the functions $x \mapsto B([\begin{smallmatrix} x & \\ & 1\end{smallmatrix}])$, where $B \in \mathcal{B}(\pi,\Lambda,\beta)$, are listed in Table 4.2. Using the fact that

$$\sum_{m=m_0}^{\infty} m^j z^m = g(z)\frac{1}{(1-z)^{j+1}} \qquad (4.2.3)$$

with a function $g(z)$ which is holomorphic and non-vanishing at $z=1$, the integrals (4.2.1) are thus easily calculated up to elements of $\mathbb{C}[q^s,q^{-s}]$. □

Also indicated in Table 4.5 are the generic representations (i.e., those that admit a Whittaker model); supercuspidals may or may not be generic. We see that for all generic representations $L^{\rm PS}_{\rm reg}(s,\pi,\mu) = L(s,\varphi)$ if $\mu = 1_{F^\times}$. Here $L(s,\varphi)$ is the L-factor of the Langlands parameter φ of π, as listed in Table A.8 of [10].

Theorem 4.2.3. *Table 4.6 shows the factors $L^{\rm PS}_{\rm reg}(s,\pi,\mu)$ for all irreducible, admissible representations (π,V) of $\mathrm{GSp}(4,F)$ in the split case. An entry "—" indicates that no split Bessel functional exists.*

Proof. Similar to the proof of Theorem 4.2.2, it is straightforward from Corollary 4.1.20. □

4.2.2 The group G

We now recall the setup of [9]. Let L be the quadratic extension of F. Let $V = L^2$, which we consider as a space of row vectors. We endow V with the

Table 4.5: Regular parts of Piatetski-Shapiro L-factors (non-split case).

		representation	$L^{\mathrm{PS}}_{\mathrm{reg}}(s,\pi,\mu)$	generic
I		$\chi_1 \times \chi_2 \rtimes \sigma$ (irreducible)	$L(s,\chi_1\chi_2\sigma\mu)L(s,\sigma\mu)$ $L(s,\chi_1\sigma\mu)L(s,\chi_2\sigma\mu)$	•
II	a	$\chi\mathrm{St}_{\mathrm{GL}(2)} \rtimes \sigma$	$L(s,\nu^{1/2}\chi\sigma\mu)L(s,\chi^2\sigma\mu)$ $L(s,\sigma\mu)$	•
	b	$\chi 1_{\mathrm{GL}(2)} \rtimes \sigma$	$L(s,\nu^{-1/2}\chi\sigma\mu)L(s,\chi^2\sigma\mu)$ $L(s,\sigma\mu)$	
III	a	$\chi \rtimes \sigma\mathrm{St}_{\mathrm{GSp}(2)}$	$L(s,\nu^{1/2}\chi\sigma\mu)L(s,\nu^{1/2}\sigma\mu)$	•
	b	$\chi \rtimes \sigma 1_{\mathrm{GSp}(2)}$	—	
IV	a	$\sigma\mathrm{St}_{\mathrm{GSp}(4)}$	$L(s,\nu^{3/2}\sigma\mu)$	•
	b	$L(\nu^2,\nu^{-1}\sigma\mathrm{St}_{\mathrm{GSp}(2)})$	$L(s,\nu^{3/2}\sigma\mu)L(s,\nu^{-1/2}\sigma\mu)$	
	c	$L(\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)},\nu^{-3/2}\sigma)$	—	
	d	$\sigma 1_{\mathrm{GSp}(4)}$	—	
V	a	$\delta([\xi,\nu\xi],\nu^{-1/2}\sigma)$	$L(s,\nu^{1/2}\xi\sigma\mu)L(s,\nu^{1/2}\sigma\mu)$	•
	b	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma)$	$L(s,\nu^{1/2}\xi\sigma\mu)L(s,\nu^{-1/2}\sigma\mu)$	
	c	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\xi\sigma)$	$L(s,\nu^{-1/2}\xi\sigma\mu)L(s,\nu^{1/2}\sigma\mu)$	
	d	$L(\nu\xi,\xi \rtimes \nu^{-1/2}\sigma)$	$L(s,\nu^{-1/2}\xi\sigma\mu)L(s,\nu^{-1/2}\sigma\mu)$	
VI	a	$\tau(S,\nu^{-1/2}\sigma)$	$L(s,\nu^{1/2}\sigma\mu)^2$	•
	b	$\tau(T,\nu^{-1/2}\sigma)$	$L(s,\nu^{1/2}\sigma\mu)$	
	c	$L(\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma)$	—	
	d	$L(\nu,1_{F^\times} \rtimes \nu^{-1/2}\sigma)$	—	
VII		$\chi \rtimes \pi$	1	•

		representation	$L^{\mathrm{PS}}_{\mathrm{reg}}(s,\pi,\mu)$	generic
VIII	a	$\tau(S,\pi)$	1	•
	b	$\tau(T,\pi)$	1	
IX	a	$\delta(\nu\xi,\nu^{-1/2}\pi(\mu))$	1	•
	b	$L(\nu\xi,\nu^{-1/2}\pi(\mu))$	1	
X		$\pi\rtimes\sigma$	$L(s,\omega_\pi\sigma\mu)L(s,\sigma\mu)$	•
XI	a	$\delta(\nu^{1/2}\pi,\nu^{-1/2}\sigma)$	$L(s,\nu^{1/2}\sigma\mu)$	•
	b	$L(\nu^{1/2}\pi,\nu^{-1/2}\sigma)$	$L(s,\nu^{-1/2}\sigma\mu)$	
		supercuspidal	1	○

skew-symmetric F-linear form

$$\rho(x,y)=\mathrm{Tr}_{L/F}(x_1y_2-x_2y_1),\qquad x=(x_1,x_2),\ y=(y_1,y_2).\tag{4.2.4}$$

Let

$$\mathrm{GSp}_\rho=\{g\in\mathrm{GL}(4,F):\rho(xg,yg)=\lambda\rho(x,y),\text{some }\lambda=\lambda(g)\in F^\times,\text{for all }x,y\in V\}$$

be the symplectic similitude group of the form ρ. Let

$$G=\{g\in\mathrm{GL}(2,L):\ \det(g)\in F^\times\}.\tag{4.2.5}$$

The group G acts on V by matrix multiplication from the right. A calculation shows that

$$\rho(xg,yg)=\det(g)\rho(x,y)\tag{4.2.6}$$

Table 4.6: Regular parts of Piatetski-Shapiro L-factors (split case).

		representation	$L^{\rm PS}_{\rm reg}(s,\pi,\mu)$	generic
I		$\chi_1\times\chi_2\rtimes\sigma$ (irreducible)	$L(s,\chi_1\chi_2\sigma\mu)L(s,\sigma\mu)$ $L(s,\chi_1\sigma\mu)L(s,\chi_2\sigma\mu)$	•
II	a	$\chi\mathrm{St}_{\mathrm{GL}(2)}\rtimes\sigma$	$L(s,\nu^{1/2}\chi\sigma\mu)L(s,\chi^2\sigma\mu)$ $L(s,\sigma\mu)$	•
	b	$\chi 1_{\mathrm{GL}(2)}\rtimes\sigma$	$L(s,\nu^{-1/2}\chi\sigma\mu)L(s,\chi^2\sigma\mu)$ $L(s,\sigma\mu)$	
III	a	$\chi\rtimes\sigma\mathrm{St}_{\mathrm{GSp}(2)}$	$L(s,\nu^{1/2}\chi\sigma\mu)L(s,\nu^{1/2}\sigma\mu)$	•
	b	$\chi\rtimes\sigma 1_{\mathrm{GSp}(2)}$	$L(s,\nu^{-1/2}\chi\sigma\mu)L(s,\nu^{-1/2}\sigma\mu)$	
IV	a	$\sigma\mathrm{St}_{\mathrm{GSp}(4)}$	$L(s,\nu^{3/2}\sigma\mu)$	•
	b	$L(\nu^2,\nu^{-1}\sigma\mathrm{St}_{\mathrm{GSp}(2)})$	$L(s,\nu^{3/2}\sigma\mu)L(s,\nu^{-1/2}\sigma\mu)$	
	c	$L(\nu^{3/2}\mathrm{St}_{\mathrm{GL}(2)},\nu^{-3/2}\sigma)$	$L(s,\nu^{-3/2}\sigma\mu)L(s,\nu^{1/2}\sigma\mu)$	
	d	$\sigma 1_{\mathrm{GSp}(4)}$	—	
V	a	$\delta([\xi,\nu\xi],\nu^{-1/2}\sigma)$	$L(s,\nu^{1/2}\xi\sigma\mu)L(s,\nu^{1/2}\sigma\mu)$	•
	b	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma)$	$L(s,\nu^{1/2}\xi\sigma\mu)L(s,\nu^{-1/2}\sigma\mu)$	
	c	$L(\nu^{1/2}\xi\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\xi\sigma)$	$L(s,\nu^{-1/2}\xi\sigma\mu)L(s,\nu^{1/2}\sigma\mu)$	
	d	$L(\nu\xi,\xi\rtimes\nu^{-1/2}\sigma)$	—	
VI	a	$\tau(S,\nu^{-1/2}\sigma)$	$L(s,\nu^{1/2}\sigma\mu)^2$	•
	b	$\tau(T,\nu^{-1/2}\sigma)$	—	
	c	$L(\nu^{1/2}\mathrm{St}_{\mathrm{GL}(2)},\nu^{-1/2}\sigma)$	$L(s,\nu^{-1/2}\sigma\mu)$	
	d	$L(\nu,1_{F^\times}\rtimes\nu^{-1/2}\sigma)$	$L(s,\nu^{-1/2}\sigma\mu)^2$	
VII		$\chi\rtimes\pi$	1	•

		representation	$L^{\mathrm{PS}}_{\mathrm{reg}}(s,\pi,\mu)$	generic
VIII	a	$\tau(S,\pi)$	1	$\bullet$
	b	$\tau(T,\pi)$	1	
IX	a	$\delta(\nu\xi,\nu^{-1/2}\pi(\mu))$	1	$\bullet$
	b	$L(\nu\xi,\nu^{-1/2}\pi(\mu))$	—	
X		$\pi \rtimes \sigma$	$L(s,\omega_\pi\sigma\mu)L(s,\sigma\mu)$	$\bullet$
XI	a	$\delta(\nu^{1/2}\pi,\nu^{-1/2}\sigma)$	$L(s,\nu^{1/2}\sigma\mu)$	$\bullet$
	b	$L(\nu^{1/2}\pi,\nu^{-1/2}\sigma)$	$L(s,\nu^{-1/2}\sigma\mu)$	
		supercuspidal	1	$\circ$

for $x,y\in V$ and $g\in G$. Hence, $G\subset \mathrm{GSp}_\rho$. Since all 4-dimensional symplectic F-spaces are isomorphic to the standard space F^4 with the form (2.1.1), the groups GSp_ρ and $\mathrm{GSp}(4,F)$ are isomorphic; here, we think of $\mathrm{GSp}(4,F)$ as acting on the right on the space of row vectors F^4. We wish to find one such isomorphism under which the group G takes on a particularly simple shape inside $\mathrm{GSp}(4,F)$.

For this we assume that the matrix β in (2.1.4) is diagonal and non-degenerate, i.e., $b=0$ and $a,c\neq 0$; after a suitable conjugation, every non-degenerate β can be brought into this form. Consider the following F-basis of V,

$$f_1=(1,0),\qquad f_2=(\Delta/c,0),\qquad f_3=(0,1/2),\qquad f_4=(0,c/(2\Delta)). \quad (4.2.7)$$

Let $e_1,\ldots,e_4$ be the standard basis of F^4. Then the map $f_i\mapsto e_i$ establishes an isomorphism $V\cong F^4$ preserving the symplectic form on both spaces (the form ρ on V, and the form J defined in (2.1.1) on F^4). The resulting isomorphism

$\mathrm{GSp}_\rho \cong \mathrm{GSp}(4, F)$ has the following properties,

$$G \ni \begin{bmatrix} x & \\ & 1 \end{bmatrix} \longmapsto \begin{bmatrix} x & & & \\ & x & & \\ & & 1 & \\ & & & 1 \end{bmatrix}, \tag{4.2.8}$$

$$G \ni \begin{bmatrix} 1 & \\ & x \end{bmatrix} \longmapsto \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & x & \\ & & & x \end{bmatrix}, \tag{4.2.9}$$

$$G \ni \begin{bmatrix} t & \\ & \bar{t} \end{bmatrix} \longmapsto \begin{bmatrix} x & yc & & \\ -ya & x & & \\ & & x & ya \\ & & -yc & x \end{bmatrix} \qquad \text{for } t = x + y\Delta \in L^\times, \tag{4.2.10}$$

$$G \ni \begin{bmatrix} 1 & x+y\Delta \\ & 1 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & & 2x & -2ay \\ & 1 & -2ay & -2ac^{-1}x \\ & & 1 & \\ & & & 1 \end{bmatrix}. \tag{4.2.11}$$

Here, $\bar{t} = x - y\Delta$ is the Galois conjugate of t. Recall from (2.1.9) that the matrices on the right hand side of (4.2.10) are precisely the elements of T. It is easy to verify that the matrices on the right hand side of (4.2.11) are precisely those elements of N that lie in

$$N_0 = \{\begin{bmatrix} 1 & X \\ & 1 \end{bmatrix} : \operatorname{tr}(\beta X) = 0\} = \{\begin{bmatrix} 1 & & x & y \\ & 1 & y & z \\ & & 1 & \\ & & & 1 \end{bmatrix} : ax + by + cz = 0\}. \tag{4.2.12}$$

In particular, if we consider G a subgroup of $\mathrm{GSp}(4, F)$, then we see that

$$G \cap R = TN_0;$$

see Proposition 2.1 of [9]. We define the following subgroups of G,

$$A^G = G \cap \begin{bmatrix} * & \\ & * \end{bmatrix} = \{\begin{bmatrix} xt & \\ & \bar{t} \end{bmatrix} \in \mathrm{GL}(2, L) \ : \ x \in F^\times,\ t \in L^\times\}, \tag{4.2.13}$$

$$N_0 = G \cap \begin{bmatrix} 1 & * \\ & 1 \end{bmatrix} = \{\begin{bmatrix} 1 & b \\ & 1 \end{bmatrix} \in \mathrm{GL}(2, L) \ : \ b \in L\}, \tag{4.2.14}$$

$$B^G = G \cap \begin{bmatrix} * & * \\ & * \end{bmatrix} = \{\begin{bmatrix} a & b \\ & d \end{bmatrix} \in \mathrm{GL}(2, L) \ : \ ad \in F^\times\}, \tag{4.2.15}$$

$$K^G = G \cap \mathrm{GL}(2, \mathfrak{o}_L) = \{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}(2, \mathfrak{o}_L) \ : \ ad - bc \in F^\times\}. \tag{4.2.16}$$

By our remarks above, when embedded into $\mathrm{GSp}(4, F)$, the group N_0 coincides with the group introduced in (4.2.12), so that the notation is consistent. The Iwasawa decomposition for $\mathrm{GL}(2, L)$ implies that $G = B^G K^G$. The modular factor for B^G is $\delta([\begin{smallmatrix} a & b \\ & d \end{smallmatrix}]) = |a/d|_L$, where $|\cdot|_L$ is the normalized absolute value on L. Note that $|t|_L = |N_{L/F}(t)|_F$ for $t \in L^\times$. Let dn be the Haar measure on N_0 that gives $N_0 \cap K^G$ volume 1. Let da be the Haar measure on A^G that gives $A^G \cap K^G$ volume 1. Let dk be the Haar measure on K^G with total volume 1. There is a Haar measure on G given by

$$\int\limits_{N_0} \int\limits_{A^G} \int\limits_{K^G} f(nak)\delta(a)^{-1}\, dk\, da\, dn. \tag{4.2.17}$$

The measure (4.2.17) gives K^G volume 1. We will also use the integration formula

$$\int\limits_{N_0\backslash G} f(g)\, dg = \int\limits_{B^G} f(wb)\, db = \int\limits_{N_0} \int\limits_{A^G} f(wna)\, da\, dn \tag{4.2.18}$$

for a function f on G that is left N_0-invariant (the db in the middle integral is a *right* Haar measure on B^G). Here, $w = [\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}] \in G$, which is embedded into $\mathrm{GSp}(4, F)$ as

$$w \longmapsto \begin{bmatrix} 2 &&& \\ & -2ac^{-1} && \\ && \frac{1}{2} & \\ &&& -\frac{1}{2}ca^{-1} \end{bmatrix} \begin{bmatrix} && 1 & \\ &&& 1 \\ -1 &&& \\ & -1 && \end{bmatrix}. \tag{4.2.19}$$

Principal series representations of G

Let Λ be a character of $L^\times$, let μ be a character of $F^\times$, and $s \in \mathbb{C}$. We denote by $\mathcal{J}(\Lambda, \mu, s)$ the induced representation $\mathrm{ind}_{B^G}^G(\chi)$ (unnormalized induction), where

$$\chi([\begin{smallmatrix} xt & * \\ & \bar{t} \end{smallmatrix}]) = \mu(x)|x|^{s+1/2}\Lambda(t)^{-1}. \tag{4.2.20}$$

It is easy to see that the contragredient of $\mathcal{J}(\Lambda, \mu, s)$ is $\mathcal{J}(\Lambda^{-1}, \mu^{-1}, 1-s)$.

Let $V = L^2$, considered as a space of row vectors. Let $\mathcal{S}(V)$ be the space of Schwartz-Bruhat functions on V, i.e., the space of locally constant functions with compact support. For $g \in G$, $\Phi \in \mathcal{S}(V)$ and a complex number s, we define

$$f^{\Phi}(g, \mu, \Lambda, s) := \mu(\det(g))|\det(g)|^{s+1/2} \int\limits_{L^{\times}} \Phi((0, \bar{t})g)|t\bar{t}|^{s+1/2}\mu(t\bar{t})\Lambda(t)\, d^{\times}t. \tag{4.2.21}$$

This is the same definition as on p. 265 of [9], except we have $(0, \bar{t})$ instead of $(0, t)$, in order to be compatible with our conventions about Bessel models. Assuming convergence, a calculation shows that $f^{\Phi} \in \mathcal{J}(\Lambda, \mu, s)$.

Let $\mathcal{S}_0(V)$ be the subspace of $\Phi \in \mathcal{S}(V)$ for which $\Phi(0, 0) = 0$. If $\Phi \in \mathcal{S}_0(V)$ and $g \in G$, then $\Phi((0, \bar{t})g) = 0$ for t outside a compact set of $L^{\times}$. It follows that the integral (4.2.21) converges absolutely for $\Phi \in \mathcal{S}_0(V)$, for any $s \in \mathbb{C}$.

Lemma 4.2.4. $\mathcal{J}(\Lambda, \mu, s) = \{f^{\Phi}(\,\cdot\,, \mu, \Lambda, s) : \Phi \in \mathcal{S}_0(V)\}$.

Proof. Given $f \in \mathcal{J}(\Lambda, \mu, s)$, we need to find $\Phi \in \mathcal{S}_0(V)$ such that $f^{\Phi} = f$. We define Φ by

$$\Phi(x, y) = \begin{cases} \mu^{-1}(\det(k))f(k) & \text{if } (x, y) = (0, 1)k \text{ for some } k \in K^G, \\ 0 & \text{if } (x, y) \notin (0, 1)K^G\ . \end{cases} \tag{4.2.22}$$

It is straightforward to verify that Φ is well-defined, that $\Phi \in \mathcal{S}_0(V)$, and that f^{Φ} is a multiple of f. $\square$

Lemma 4.2.5. *Let* $\Lambda_{\mu} = \Lambda \cdot (\mu \circ N_{L/F})$.

i) The representation $\mathcal{J}(\Lambda, \mu, s)$ *contains a one-dimensional G-invariant subspace if and only if*

$$\Lambda_\mu(t) = |t|_L^{-s-1/2} \qquad \textit{for all } t \in L^\times. \tag{4.2.23}$$

In this case the function

$$f(g) = \mu(\det(g))|\det(g)|^{s+1/2}, \qquad g \in G, \tag{4.2.24}$$

spans a one-dimensional G-invariant subspace of $\mathrm{ind}_{BG}^G(\chi)$.

ii) The representation $\mathcal{J}(\Lambda, \mu, s)$ *contains a one-dimensional G-invariant quotient if and only if*

$$\Lambda_\mu(t) = |t|_L^{-s+3/2} \qquad \textit{for all } t \in L^\times. \tag{4.2.25}$$

Proof. Part i) is an easy exercise. Part ii) follows from i), observing that the contragredient of $\mathcal{J}(\Lambda, \mu, s)$ is $\mathcal{J}(\Lambda^{-1}, \mu^{-1}, 1-s)$. $\square$

We observe that condition (4.2.23) is equivalent to saying that s is a pole of $L(s+1/2, \Lambda_\mu)$. Later we will define the notion of *exceptional pole*; see (4.2.41). The exceptional poles will be among the poles of $L(s+1/2, \Lambda_\mu)$. Note that, by (4.2.22), the function f in (4.2.24) is a multiple of f^Φ, where

$$\Phi(x,y) = \begin{cases} 1 & \text{if } (x,y) = (0,1)k \text{ for some } k \in K^G, \\ 0 & \text{if } (x,y) \notin (0,1)K^G. \end{cases} \tag{4.2.26}$$

Hence, in the non-split case, Φ is the characteristic function of $(\mathfrak{o}_L \oplus \mathfrak{o}_L) \backslash (\mathfrak{p}_L \oplus \mathfrak{p}_L)$.

4.2.3 The zeta integrals

Let Λ be a character of $T \cong L^\times$, and let μ be a character of $F^\times$. Recall the definition of the functions $f^\Phi(g, \mu, \Lambda, s)$ in (4.2.21). Let π be an irreducible, admissible representation of $\mathrm{GSp}(4, F)$. Let $\mathcal{B}(\pi, \Lambda, \beta)$ be a (Λ, β)-Bessel model for π. For $B \in \mathcal{B}(\pi, \Lambda, \beta)$ and $s \in \mathbb{C}$, let

$$Z(s, B, \Phi, \mu) = \int\limits_{TN_0 \backslash G} B(g) f^\Phi(g, \mu, \Lambda, s)\, dg, \tag{4.2.27}$$

provided this integral converges. (In [9] this integral was denoted by $L(W, \Phi, \mu, s)$.) Substituting the definition of $f^\Phi(g, \mu, \Lambda, s)$ and unfolding the integral shows that

$$Z(s, B, \Phi, \mu) = \int\limits_{N_0 \backslash G} B(g) \Phi((0, 1)g) \mu(\det(g)) |\det(g)|^{s+1/2}\, dg. \tag{4.2.28}$$

By (4.2.17), we have

$$Z(s, B, \Phi, \mu) = \int\limits_{A^G} \int\limits_{K^G} \delta(a)^{-1} B(ak) \Phi((0, 1)ak) \mu(\det(ak)) |\det(ak)|^{s+1/2}\, dk\, da. \tag{4.2.29}$$

Recall that $\mathcal{S}_0(V)$ is the space of $\Phi \in \mathcal{S}(V)$ satisfying $\Phi(0, 0) = 0$. Let $\Phi_1 \in \mathcal{S}(V)$ be the characteristic function of $\mathfrak{o}_L \oplus \mathfrak{o}_L$. Then every $\Phi \in \mathcal{S}(V)$ can be written in a unique way as $\Phi = \Phi_0 + c\Phi_1$ with $\Phi_0 \in \mathcal{S}_0(V)$ and $c \in \mathbb{C}$. We will first investigate $Z(s, B, \Phi, \mu)$ for $\Phi \in \mathcal{S}_0(V)$.

Lemma 4.2.6. *Let the notations and hypotheses be as above.*

i) For any $B \in \mathcal{B}(\pi, \Lambda, \beta)$ and $\Phi \in \mathcal{S}_0(V)$, the function $Z(s, B, \Phi, \mu)$ converges for real part of s large enough to an element of $\mathbb{C}(q^{-s})$. This element lies in the ideal $I(\pi, \mu)$ generated by all simplified zeta integrals; see Proposition 4.2.1.

ii) For any $B \in \mathcal{B}(\pi, \Lambda, \beta)$, there exists $\Phi \in \mathcal{S}_0(V)$ such that $Z(s, B, \Phi, \mu) = \zeta(s, B, \mu)$.

Hence, the integrals $Z(s, B, \Phi, \mu)$, as B runs through $\mathcal{B}(\pi, \Lambda, \beta)$ and Φ runs through $\mathcal{S}_0(V)$, generate the ideal $I(\pi, \mu)$ already exhibited in Proposition 4.2.1.

Proof. i) Let $\Phi \in \mathcal{S}_0(V)$. We have

$$\Phi((0,1)ak) = \Phi(\bar{t}k_3, \bar{t}k_4) \qquad \text{if } a = \left[\begin{smallmatrix} xt & \\ & \bar{t} \end{smallmatrix}\right] \in A^G,\ k = \left[\begin{smallmatrix} k_1 & k_2 \\ k_3 & k_4 \end{smallmatrix}\right] \in K^G. \tag{4.2.30}$$

Since one of k_3 or k_4 is a unit and $\Phi(0,0) = 0$, it follows that $\Phi((0,1)ak) = 0$ if t is outside a compact set of $L^\times$. As a consequence, there exists a small subgroup Γ of K^G such that $\Phi((0,1)ak\gamma) = \Phi((0,1)ak)$ for all $a \in A^G$, $k \in K^G$ and $\gamma \in \Gamma$. By making Γ even smaller, we may assume that B and $\mu \circ \det$ are right Γ-invariant. It follows that $Z(s, B, \Phi, \mu)$ as in (4.2.29) is a finite sum of integrals of the form

$$I(s, B, \Phi, \mu) = \int\limits_{A^G} \delta(a)^{-1} B(a) \Phi((0,1)a) \mu(\det(a)) |\det(a)|^{s+1/2}\, da, \tag{4.2.31}$$

with different B and $\Phi \in \mathcal{S}_0(V)$. Using coordinates on A^G, we have

$$I(s, B, \Phi, \mu) = \int\limits_{F^\times} \int\limits_{L^\times} |xt\bar{t}^{-1}|_L^{-1} B(\left[\begin{smallmatrix} xt & \\ & \bar{t} \end{smallmatrix}\right]) \Phi(0, \bar{t}) \mu(xt\bar{t}) |xt\bar{t}|^{s+1/2}\, d^\times t\, d^\times x$$

$$= \int_{F^\times}\int_{L^\times} |x|^{-2}\Lambda(t)B([\begin{smallmatrix} x & \\ & 1\end{smallmatrix}])\Phi(0,\bar t)\mu(xt\bar t)|xt\bar t|^{s+1/2}\, d^\times t\, d^\times x$$

$$= \Big(\int_{F^\times} B([\begin{smallmatrix} x & \\ & 1\end{smallmatrix}])\mu(x)|x|^{s-3/2}\, d^\times x\Big)\Big(\int_{L^\times} \Lambda(t)\Phi(0,\bar t)\mu(t\bar t)|t\bar t|^{s+1/2}\, d^\times t\Big). \tag{4.2.32}$$

The first integral is precisely $\zeta(s,B,\mu)$; see (4.2.1). Since the integration in the second integral is over a compact subset of $L^\times$, this integral is in $\mathbb{C}[q^s,q^{-s}]$. It follows that $I(s,B,\Phi,\mu)$ lies in the ideal $I(\pi,\mu)$.

ii) By (4.2.28) and (4.2.18), we have

$$Z(s,B,\Phi,\mu) = \int_{N_0}\int_{A^G} B(wna)\Phi((0,1)wna)\mu(\det(a))|\det(a)|^{s+1/2}\, da\, dn$$

$$= \int_{N_0}\int_{A^G} B(wna)\Phi((-1,0)na)\mu(\det(a))|\det(a)|^{s+1/2}\, da\, dn$$

$$= \int_{L}\int_{F^\times}\int_{L^\times} B(w[\begin{smallmatrix} 1 & n\\ & 1\end{smallmatrix}][\begin{smallmatrix} xt & \\ & \bar t\end{smallmatrix}])\Phi((-1,0)[\begin{smallmatrix} 1 & n\\ & 1\end{smallmatrix}][\begin{smallmatrix} xt & \\ & \bar t\end{smallmatrix}])$$

$$\mu(xt\bar t)|xt\bar t|^{s+1/2}\, d^\times t\, d^\times x\, dn$$

$$= \int_{L}\int_{F^\times}\int_{L^\times} B(w\big[\begin{smallmatrix} xt & \bar t n\\ & \bar t\end{smallmatrix}\big])\Phi(-xt,-\bar t n)\mu(xt\bar t)|x|^{s+1/2}|t|_L^{s+1/2}\, d^\times t\, d^\times x\, dn$$

$$= \int_{L}\int_{F^\times}\int_{L^\times} B(w[\begin{smallmatrix} xt & n\\ & \bar t\end{smallmatrix}])\Phi(-xt,-n)\mu(xt\bar t)|x|^{s+1/2}|t|_L^{s-1/2}\, d^\times t\, d^\times x\, dn$$

$$= \int_{L}\int_{F^\times}\int_{L^\times} B(w\big[\begin{smallmatrix} 1 & \\ & x^{-1}\end{smallmatrix}\big][\begin{smallmatrix} t & n\\ & \bar t\end{smallmatrix}])\Phi(-t,-n)$$

$$\mu(x)^{-1}\mu(t\bar t)|x|^{3/2-s}|t|_L^{s-1/2}\, d^\times t\, d^\times x\, dn.$$

Now choose Φ such that $\Phi(-t,-n)$ is zero unless t is close to 1 and n is close to 0.

If the support of Φ is chosen small enough, then, after appropriate normalization,

$$Z(s,B,\Phi,\mu) = \int\limits_{F^\times} B(\left[\begin{smallmatrix} x^{-1} & \\ & 1 \end{smallmatrix}\right] w)\mu(x)^{-1}|x|^{3/2-s}\, d^\times x.$$

This is just $\zeta(s,wB,\mu)$. The assertion follows. $\square$

We see from Lemma 4.2.6 that, instead of (4.2.2), we could have defined $L^{\mathrm{PS}}_{\mathrm{reg}}(s,\pi,\mu)$ as the gcd of all $Z(s,B,\Phi,\mu)$, as B runs through $\mathcal{B}(\pi,\Lambda,\beta)$ and Φ runs through $\mathcal{S}_0(V)$. The same observation was made in Proposition 18 i) of [6].

Next we investigate $Z(s,B,\Phi_1,\mu)$, where we recall Φ_1 is the characteristic function of $\mathfrak{o}_L \oplus \mathfrak{o}_L$. In the split case, a character Λ of $L^\times = F^\times \times F^\times$ is a pair (λ_1,λ_2) of characters of $F^\times$, and by $L(s,\Lambda)$ we mean $L(s,\lambda_1)L(s,\lambda_2)$.

Lemma 4.2.7. *Let $\Lambda_\mu = \Lambda\cdot(\mu\circ N_{L/F})$.*

i) Assume that Λ_μ is ramified. Then $Z(s,B,\Phi_1,\mu)=0$.

ii) Assume that Λ_μ is unramified. Then

$$Z(s,B,\Phi_1,\mu) = \zeta(s,B_\mu,\mu)L(s+1/2,\Lambda_\mu), \tag{4.2.33}$$

where

$$B_\mu(g) := \int\limits_{K^G} B(gk)\mu(\det(k))\,dk, \qquad g\in \mathrm{GSp}(4,F). \tag{4.2.34}$$

Proof. Evidently, $\Phi_1((x,y)k)=\Phi_1(x,y)$ for all $(x,y)\in V$ and $k\in K^G$. Therefore, from (4.2.29), we get

$$Z(s,B,\Phi_1,\mu) = \int\limits_{A^G}\int\limits_{K^G} \delta(a)^{-1}B(ak)\Phi_1((0,1)a)\mu(\det(ak))|\det(a)|^{s+1/2}\,dk\,da$$
$$= \int\limits_{A^G} \delta(a)^{-1}B_\mu(a)\Phi_1((0,1)a)\mu(\det(a))|\det(a)|^{s+1/2}\,da. \quad (4.2.35)$$

Clearly, B_μ is an element of $\mathcal{B}(\pi,\Lambda,\beta)$ satisfying $B_\mu(gk) = \mu^{-1}(\det(k))B_\mu(g)$ for $k \in K^G$. Using coordinates on A^G, we have

$$Z(s,B,\Phi_1,\mu) = \int\limits_{F^\times}\int\limits_{L^\times} |xt\bar{t}^{-1}|_L^{-1}B_\mu(a)\Phi_1((0,\bar{t}))\mu(xt\bar{t})|xt\bar{t}|^{s+1/2}\,d^\times t\,d^\times x$$
$$= \int\limits_{F^\times}\int\limits_{L^\times} B_\mu([\begin{smallmatrix} xt & \\ & \bar{t}\end{smallmatrix}])\Phi_1((0,\bar{t}))\mu(xt\bar{t})|t\bar{t}|^{s+1/2}|x|^{s-3/2}\,d^\times t\,d^\times x$$
$$= \int\limits_{F^\times}\int\limits_{L^\times\cap\mathfrak{o}_L} \Lambda(t)B_\mu([\begin{smallmatrix} x & \\ & 1\end{smallmatrix}])\mu(xt\bar{t})|t\bar{t}|^{s+1/2}|x|^{s-3/2}\,d^\times t\,d^\times x$$
$$= \zeta(s,B_\mu,\mu)\int\limits_{L^\times\cap\mathfrak{o}_L} \Lambda(t)\mu(t\bar{t})|t\bar{t}|^{s+1/2}\,d^\times t. \quad (4.2.36)$$

It is straightforward to calculate that

$$\int\limits_{L^\times\cap\mathfrak{o}_L} \Lambda(t)\mu(t\bar{t})|t\bar{t}|^{s+1/2}\,d^\times t = \begin{cases} L(s+1/2,\Lambda_\mu) & \text{if } \Lambda_\mu \text{ is unramified,} \\ 0 & \text{if } \Lambda_\mu \text{ is ramified.}\end{cases} \quad (4.2.37)$$

This concludes the proof. □

We see from Lemma 4.2.6 and Lemma 4.2.7 that $Z(s,B,\Phi,\mu)$ converges for real part of s large enough to an element of $\mathbb{C}(q^{-s})$, for any $B \in \mathcal{B}(\pi,\Lambda,\beta)$ and $\Phi \in \mathcal{S}(V)$. Let $I_{\Lambda,\beta}(\pi,\mu)$ be the $\mathbb{C}$-vector subspace of $\mathbb{C}(q^{-s})$ spanned by all

$\zeta(s, B, \mu)$ as B runs through $\mathcal{B}(\pi, \Lambda, \beta)$.

Proposition 4.2.8. *Let π be an irreducible, admissible representation of* $\mathrm{GSp}(4, F)$ *admitting a (Λ, β)-Bessel model with β as in* (2.1.4). *Then $I_{\Lambda,\beta}(\pi, \mu)$ is a non-zero $\mathbb{C}[q^{-s}, q^s]$ module containing $\mathbb{C}$, and there exists $R(X) \in \mathbb{C}[X]$ such that $R(q^{-s})I_{\Lambda,\beta}(\pi, \mu) \subset \mathbb{C}[q^{-s}, q^s]$, so that $I_{\Lambda,\beta}(\pi, \mu)$ is a fractional ideal of the principal ideal domain $\mathbb{C}[q^{-s}, q^s]$ whose quotient field is $\mathbb{C}(q^{-s})$. The fractional ideal $I_{\Lambda,\beta}(\pi, \mu)$ admits a generator of the form $1/Q(q^{-s})$ with $Q(0) = 1$, where $Q(X) \in \mathbb{C}[X]$.*

Proof. The proof is similar to that of Proposition 4.2.1. It follows easily from (4.2.28) that $I_{\Lambda,\beta}(\pi, \mu)$ is a $\mathbb{C}[q^s, q^{-s}]$-module. It follows from Lemma 4.2.6 and Proposition 4.2.1 that $I_{\Lambda,\beta}(\pi, \mu)$ contains $\mathbb{C}$. □

Using the notation of this proposition, we set

$$L_{\Lambda}^{\mathrm{PS}}(s, \pi, \mu) := 1/Q(q^{-s}). \tag{4.2.38}$$

This is the Piatetski-Shapiro L-factor, as defined in [9]. Our notation indicates that these factors may depend on Λ (and β, which we suppress from the notation).

We now distinguish two cases:

(A) Assume that

$$\frac{Z(s, B, \Phi, \mu)}{L_{\mathrm{reg}}^{\mathrm{PS}}(s, \pi, \mu)} \text{ is entire for all } B \in \mathcal{B}(\pi, \Lambda, \beta) \text{ and } \Phi \in \mathcal{S}(V). \tag{4.2.39}$$

Being entire is equivalent to lying in $\mathbb{C}[q^s, q^{-s}]$. Hence, in this case the fractional ideal generated by all $Z(s, B, \Phi, \mu)$ is generated by $L_{\mathrm{reg}}^{\mathrm{PS}}(s, \pi, \mu)$, and we have

$$L_{\Lambda}^{\mathrm{PS}}(s,\pi,\mu) = L_{\mathrm{reg}}^{\mathrm{PS}}(s,\pi,\mu). \tag{4.2.40}$$

In particular, the Piatetski-Shapiro L-factor does not depend on Λ in this case.

(B) Assume that

$$\frac{Z(s,B,\Phi,\mu)}{L_{\mathrm{reg}}^{\mathrm{PS}}(s,\pi,\mu)} \text{ has a pole for some } B \in \mathcal{B}(\pi,\Lambda,\beta) \text{ and } \Phi \in \mathcal{S}(V). \tag{4.2.41}$$

Such poles are called *exceptional poles*. By (4.2.33), exceptional poles are precisely the poles of

$$\frac{\zeta(s,B_\mu,\mu)}{L_{\mathrm{reg}}^{\mathrm{PS}}(s,\pi,\mu)} L(s+1/2,\Lambda_\mu), \tag{4.2.42}$$

as B runs through $\mathcal{B}(\pi,\Lambda,\beta)$. Since the fraction in (4.2.42) is entire, exceptional poles are found among the poles of $L(s+1/2,\Lambda_\mu)$. If we write

$$L(s,\Lambda_\mu) = \frac{1}{(1-\gamma_1 q^{-s})(1-\gamma_2 q^{-s})}, \tag{4.2.43}$$

where one of the complex numbers γ_1,γ_2 may be zero, then

$$L^{\mathrm{PS}}(s,\pi,\mu) = L_{\mathrm{reg}}^{\mathrm{PS}}(s,\pi,\mu)\frac{1}{P(q^{-s-1/2})}, \tag{4.2.44}$$

where $P \in \mathbb{C}[X]$ is either $1-\gamma_i X$ or $(1-\gamma_1 X)(1-\gamma_2 X)$.

Remark: Our definition of exceptional pole is slightly more general than the one given in [9]. According to [9], a complex number s_0 is called an exceptional pole if s_0 is a pole of $L^{\mathrm{PS}}(s,\pi,\mu)$ but not of $L^{\mathrm{PS}}_{\mathrm{reg}}(s,\pi,\mu)$. It follows easily that an exceptional pole according to [9] is also an exceptional pole according to our definition. However, the two notions may not coincide if there is overlap between the poles of $L^{\mathrm{PS}}_{\mathrm{reg}}(s,\pi,\mu)$ and the poles of $L(s+1/2,\Lambda_\mu)$.

The *regular poles* are the poles of $L^{\mathrm{PS}}_{\mathrm{reg}}(s,\pi,\mu)$. According to our definition, an exceptional pole can also be regular, while in [9] the two notions are exclusive. Our definition is designed in such a way that $L^{\mathrm{PS}}(s,\pi,\mu) \neq L^{\mathrm{PS}}_{\mathrm{reg}}(s,\pi,\mu)$ precisely if there exist exceptional poles.

4.2.4 Double coset decompositions

We first prove the following double coset decomposition for $\mathrm{GL}(2,F)$. Let β be as in (2.1.4), and let T be the group of all

$$\begin{bmatrix} x+yb/2 & yc \\ -ya & x-yb/2 \end{bmatrix} \in \mathrm{GL}(2,F), \qquad x^2 - y^2\Big(\frac{b^2}{4}-ac\Big) \neq 0. \tag{4.2.45}$$

Recall that we are in the *split case* if and only if $b^2-4ac \in F^{\times 2}$. We can and will make the assumption that

$$a, c \neq 0. \tag{4.2.46}$$

In the split case, let $r_1, r_2 \in F^\times$ be the two roots of the equation

$$ar^2 + br + c = 0. \tag{4.2.47}$$

Let B_1 be the subgroup of $\mathrm{GL}(2,F)$ consisting of all elements of the form $\left[\begin{smallmatrix} 1 & * \\ & * \end{smallmatrix}\right]$, and let B_2 be the subgroup consisting of all elements of the form $\left[\begin{smallmatrix} 1 & \\ * & * \end{smallmatrix}\right]$.

Lemma 4.2.9. *i) In the non-split case,* $\mathrm{GL}(2,F) = TB_1 = TB_2$.

ii) In the split case,

$$\begin{aligned}\mathrm{GL}(2,F) &= TB_1 \sqcup Tg_1sB_1 \sqcup Tg_2sB_1\\ &= TB_2 \sqcup Tg_1B_2 \sqcup Tg_2B_2, \qquad \textit{where } g_i = \left[\begin{smallmatrix}1 & r_i\\ & 1\end{smallmatrix}\right],\ s = \left[\begin{smallmatrix} & 1\\ -1 & \end{smallmatrix}\right].\end{aligned} \tag{4.2.48}$$

The set TB_1 (resp. TB_2) is open and dense in $\mathrm{GL}(2,F)$, and consists of all $\left[\begin{smallmatrix}a_1 & a_2\\ a_3 & a_4\end{smallmatrix}\right] \in \mathrm{GL}(2,F)$ with $aa_1^2+ba_1a_3+ca_3^2 \neq 0$ (resp. $aa_2^2+ba_2a_4+ca_4^2 \neq 0$). For $i=1$ or 2, the set Tg_isB_1 (resp. Tg_iB_2) consists of all $\left[\begin{smallmatrix}a_1 & a_2\\ a_3 & a_4\end{smallmatrix}\right] \in \mathrm{GL}(2,F)$ with $a_1 = a_3r_i$ (resp. $a_2 = a_4r_i$).

Proof. Calculations show that if $aa_1^2+ba_1a_3+ca_3^2 \neq 0$, then the equation

$$\begin{bmatrix} x+yb/2 & yc\\ -ya & x-yb/2\end{bmatrix}\left[\begin{smallmatrix}1 & z\\ & d\end{smallmatrix}\right] = \left[\begin{smallmatrix}a_1 & a_2\\ a_3 & a_4\end{smallmatrix}\right]$$

can be solved for x,y,z,d. Assume that $aa_1^2+ba_1a_3+ca_3^2 = 0$. Then $a_1 = a_3r_i$ for $i=1$ or $i=2$. Calculations show that the equation

$$\begin{bmatrix} x+yb/2 & yc\\ -ya & x-yb/2\end{bmatrix}g_is\left[\begin{smallmatrix}1 & z\\ & d\end{smallmatrix}\right] = \left[\begin{smallmatrix}a_1 & a_2\\ a_3 & a_4\end{smallmatrix}\right]$$

can be solved for x,y,z,d. This proves the statements for B_1, and the proof for B_2 is similar. □

Let P be the (F-points of the) Siegel parabolic subgroup of $\mathrm{GSp}(4,F)$; see (2.1.2). Let G be the group defined in (4.2.5). We assume that $\beta = \left[\begin{smallmatrix}a & \\ & c\end{smallmatrix}\right]$ with $ac \neq 0$, and embed G into $\mathrm{GSp}(4,F)$ such that (4.2.8) – (4.2.11) holds. More generally, if

$$g = \left[\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right] \in G,$$

then a calculation shows that, as an element of $\mathrm{GSp}(4,F)$,

$$g = \begin{bmatrix} \alpha_1 & c\alpha_2 & 2\beta_1 & -2a\beta_2 \\ -a\alpha_2 & \alpha_1 & -2a\beta_2 & -\frac{2a}{c}\beta_1 \\ \frac{1}{2}\gamma_1 & \frac{c}{2}\gamma_2 & \delta_1 & -a\delta_2 \\ \frac{c}{2}\gamma_2 & -\frac{c}{2a}\gamma_1 & c\delta_2 & \delta_1 \end{bmatrix}. \tag{4.2.49}$$

Here, $\alpha = \alpha_1 + \Delta\alpha_2$ etc, with Δ as defined after (2.1.7). The following result is a more precise version of a remark made in the proof of Theorem 4.3 of [9].

Lemma 4.2.10. *Assume the above notations and hypotheses. Let*

$$s_2 = \begin{bmatrix} & & 1 & \\ & 1 & & \\ -1 & & & \\ & & & 1 \end{bmatrix}. \tag{4.2.50}$$

Then

$$\mathrm{GSp}(4,F) = GP \sqcup Gs_2P. \tag{4.2.51}$$

The double coset Gs_2P is open and dense in $\mathrm{GSp}(4,F)$, *and*

$$s_2^{-1}Gs_2 \cap P = \{ \begin{bmatrix} A & \\ & \det(A)\,{}^tA^{-1} \end{bmatrix} :\ A \in \mathrm{GL}(2,F)\}. \tag{4.2.52}$$

We have $Gs_2P = Gs_2HN$, where H and N are defined in (2.1.3) *and* (2.1.2), *respectively. Furthermore,*

$$GP = \begin{cases} GB_2N & \text{in the non-split case,} \\ GB_2N \sqcup Gg_1B_2N \sqcup Gg_2B_2N & \text{in the split case,} \end{cases} \tag{4.2.53}$$

where

$$B_2 = \{ \begin{bmatrix} 1 & & & \\ x & y & & \\ & & y & -x \\ & & & 1 \end{bmatrix} :\ x \in F,\ y \in F^\times\}, \qquad g_i = \begin{bmatrix} 1 & r_i & & \\ & 1 & & \\ & & 1 & \\ & & -r_i & 1 \end{bmatrix}, \tag{4.2.54}$$

with $r_1, r_2 \in F^\times$ *being the two roots of the equation* $ar^2 + c = 0$.

Proof. Using the description (4.2.49) of the elements of G, it is easy to verify (4.2.52). As a consequence, $Gs_2P = Gs_2HN$. Equation (4.2.53) follows from (4.2.48); the disjointness in the split case is easy to verify.

By the Bruhat decomposition,

$$\mathrm{GSp}(4,F) = P \sqcup \begin{bmatrix} 1 & & * & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} s_2P \sqcup \begin{bmatrix} 1 & & & \\ * & 1 & & * \\ & & 1 & * \\ & & & 1 \end{bmatrix} s_1s_2P \sqcup \begin{bmatrix} 1 & & * & * \\ & 1 & * & * \\ & & 1 & \\ & & & 1 \end{bmatrix} s_2s_1s_2P. \tag{4.2.55}$$

Calculations show that

$$Gs_2P \cap \begin{bmatrix} 1 & & * & * \\ & 1 & * & * \\ & & 1 & \\ & & & 1 \end{bmatrix} s_2s_1s_2P = \{[\begin{smallmatrix} 1 & X \\ & 1 \end{smallmatrix}]s_2s_1s_2p \,:\, p \in P,\ \mathrm{tr}(\beta X) \neq 0\}, \tag{4.2.56}$$

$$Gs_2P \cap \begin{bmatrix} 1 & & & \\ * & 1 & & * \\ & & 1 & * \\ & & & 1 \end{bmatrix} s_1s_2P = \{\begin{bmatrix} 1 & & & \\ x & 1 & & z \\ & & 1 & -x \\ & & & 1 \end{bmatrix} s_1s_2p \,:\, p \in P,\ x^2 \neq -a/c\}, \tag{4.2.57}$$

$$Gs_2P \cap \begin{bmatrix} 1 & & * & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} s_2P = \begin{bmatrix} 1 & & * & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} s_2P, \tag{4.2.58}$$

$$Gs_2P \cap P = \emptyset, \tag{4.2.59}$$

and

$$GP \cap \begin{bmatrix} 1 & & * & * \\ & 1 & * & * \\ & & 1 & \\ & & & 1 \end{bmatrix} s_2s_1s_2P = \{[\begin{smallmatrix} 1 & X \\ & 1 \end{smallmatrix}]s_2s_1s_2p \,:\, p \in P,\ \mathrm{tr}(\beta X) = 0\}, \tag{4.2.60}$$

$$GP \cap \begin{bmatrix} 1 & & & \\ * & 1 & & * \\ & & 1 & * \\ & & & 1 \end{bmatrix} s_1s_2P = \{\begin{bmatrix} 1 & & & \\ x & 1 & & z \\ & & 1 & -x \\ & & & 1 \end{bmatrix} s_1s_2p \,:\, p \in P,\ x^2 = -a/c\}, \tag{4.2.61}$$

$$GP \cap \begin{bmatrix} 1 & & * & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} s_2P = \emptyset, \tag{4.2.62}$$

$$GP \cap P = P. \tag{4.2.63}$$

It follows that $\mathrm{GSp}(4,F) = GP \sqcup Gs_2P$. Since the big Bruhat cell is dense in $\mathrm{GSp}(4,F)$, equation (4.2.56) implies that Gs_2P is also dense in $\mathrm{GSp}(4,F)$. Since

$GP = K^G B^G P = K^G P$ is the product of a compact and a closed set, it is closed in $\mathrm{GSp}(4, F)$. $\square$

In the proof of the following lemma we will make use of the fact that a continuous bijection $X \to Y$ between p-adic spaces is a homeomorphism. This is because we can cover X with open-compact subsets, and a continuous bijection from a compact topological space to a Hausdorff space is a homeomorphism.

For a locally compact, totally disconnected space X, we denote by $\mathcal{S}(X)$ the space of locally constant functions $X \to \mathbb{C}$ with compact support. If X is a group, $h \in X$ and $\phi \in \mathcal{S}(X)$, we denote by $R_h\phi$ the element of $\mathcal{S}(X)$ given by $x \mapsto \phi(xh)$, and by $L_h\phi$ the element of $\mathcal{S}(X)$ given by $x \mapsto \phi(h^{-1}x)$.

Let U be the unipotent radical of the Borel subgroup of $\mathrm{GSp}(4, F)$. Then U consists of all matrices of the form

$$\begin{bmatrix} 1 & & * & * \\ * & 1 & * & * \\ & & 1 & * \\ & & & 1 \end{bmatrix}$$

in $\mathrm{GSp}(4, F)$. For $c_1, c_2 \in F$, we define a character ψ_{c_1,c_2} of U by

$$\psi_{c_1,c_2}(\begin{bmatrix} 1 & & y & * \\ x & 1 & * & * \\ & & 1 & -x \\ & & & 1 \end{bmatrix}) = \psi(c_1 x + c_2 y). \tag{4.2.64}$$

The statement of the following result was mentioned in the proof of Theorem 4.3 of [9].

Lemma 4.2.11. *Let $D : \mathcal{S}(\mathrm{GSp}(4, F)) \to \mathbb{C}$ be a distribution on $\mathrm{GSp}(4, F)$ with the following properties:*

i) There exist $c_1, c_2 \in F^\times$ *such that*

$$D(R_u\phi) = \psi_{c_1,c_2}(u)D(\phi) \qquad \textit{for all } u \in U \tag{4.2.65}$$

and all $\phi \in \mathcal{S}(\mathrm{GSp}(4, F))$.

ii) There exists a character β *of* G *such that*

$$D(L_h\phi) = \beta(h)D(\phi) \qquad \textit{for all } h \in G \tag{4.2.66}$$

and all $\phi \in \mathcal{S}(\mathrm{GSp}(4, F))$.

Then $D = 0$.

Proof. Since $\mathrm{GSp}(4, F) = GP \sqcup Gs_2P$, it suffices to show that a distribution on $\mathcal{S}(Gs_2P)$ with the properties (4.2.65) and (4.2.66) is zero, and a distribution on $\mathcal{S}(GP)$ with the properties (4.2.65) and (4.2.66) is also zero.

1) First we prove that a distribution D on Gs_2P with the properties (4.2.65) and (4.2.66) must be zero. For $x \in F^\times$, let $h_x = \mathrm{diag}(x, x, 1, 1)$. By Lemma 4.2.10, $Gs_2P = Gs_2HN$. In fact, every element of Gs_2P can be written in the form gs_2h_xn with $g \in G$ and uniquely determined $x \in F^\times$ and $n \in N$. Hence Gs_2P is homeomorphic to $G \times H \times N$. We consider the continuous map

$$p: \; Gs_2P \longrightarrow F^\times \quad \text{defined by} \quad gs_2h_xn \longmapsto x.$$

The set Gs_2P is invariant under the left action of G and the right action of U. It is easy to see that every fiber $p^{-1}(x)$ is $G \times U$-invariant. By Corollary 2.1 of [1], Bernstein's Localization Principle, it is sufficient to prove that any distribution D

on $\mathcal{S}(p^{-1}(x))$ with the properties (4.2.65) and (4.2.66) vanishes, for all $x \in F^\times$.

We apply Proposition 4.3.2 of [3] with

$$G \times N \cong G s_2 h_x N = p^{-1}(x).$$

It shows that there exists a constant $c_1 \in \mathbb{C}$ such that

$$D(\phi) = c_1 \int\limits_G \int\limits_N \beta(g)\, \psi^{-1}_{c_1,c_2}(n)\, \phi(g s_2 h_x n)\, dn\, dg$$

for all $\phi \in \mathcal{S}(p^{-1}(x))$. We may choose some $z \in F$ such that

$$\psi_{c_1,c_2}(u_z) \neq 1 \qquad \text{for } u_z = \begin{bmatrix} 1 & & & \\ z & 1 & & \\ & & 1 & -z \\ & & & 1 \end{bmatrix}.$$

By (4.2.11),

$$n_z := s_2 u_z s_2^{-1} = \begin{bmatrix} 1 & & & -z \\ & 1 & -z & \\ & & 1 & \\ & & & 1 \end{bmatrix} \in N_0 \subset G,$$

so that $D(L_{n_z^{-1}}\phi) = \beta(n_z^{-1})D(\phi) = D(\phi)$ by (4.2.66). On the other hand, the substitution $g \mapsto n_z^{-1} g n_z$ shows that

$$\begin{aligned} D(L_{n_z^{-1}}\phi) &= c_1 \int\limits_G \int\limits_N \phi(n_z g s_2 h_x n)\beta(g)\psi^{-1}_{c_1,c_2}(n) dn dg \\ &= c_1 \int\limits_G \int\limits_N \phi(g n_z s_2 h_x n)\beta(g)\psi^{-1}_{c_1,c_2}(n) dn dg \\ &= c_1 \int\limits_G \int\limits_N \Phi(g s_2 u_z h_x n)\beta(g)\psi^{-1}_{c_1,c_2}(n) dn dg \\ &= c_1 \int\limits_G \int\limits_N \Phi(g s_2 h_x n u_z)\beta(g)\psi^{-1}_{c_1,c_2}(n) dn dg \end{aligned}$$

$$= \psi_{c_1,c_2}(u_z) c_1 \int\limits_G \int\limits_N \Phi(g s_2 h_x n)\beta(g)\psi_{c_1,c_2}^{-1}(n)\,dn\,dg.$$

In the last step we used (4.2.65). Hence $D(\phi) = \psi_{c_1,c_2}(u_z)D(\phi)$, which implies $D = 0$ on $\mathcal{S}(p^{-1}(x))$.

2) Next, using the decomposition (4.2.53), we prove that a distribution D on GP with the properties (4.2.65) and (4.2.66) must be zero.

2.1) We will first show that a distribution D on GB_2N with the properties (4.2.65) and (4.2.66) must be zero. We define the groups

$$H_1 := \{k_x = \begin{bmatrix} 1 &&& \\ & x && \\ && x & \\ &&& 1 \end{bmatrix} : x \in F^\times\}, \qquad U_1 := \begin{bmatrix} 1 &&& \\ * & 1 && * \\ && 1 & * \\ &&& 1 \end{bmatrix} \cap \mathrm{GSp}(4,F). \tag{4.2.67}$$

Then, with N_0 as in (4.2.12),

$$GB_2N = GUH_1 = GN_0U_1H_1 = GU_1H_1 = GH_1U_1. \tag{4.2.68}$$

In fact, it is not difficult to see that any element of GP can be written in the form gk_xu with uniquely determined $g \in G$, $x \in F^\times$ and $u \in U_1$. Hence GB_2N is homeomorphic to $G \times H_1 \times U_1$. We consider the continuous map

$$p: \; GB_2N \longrightarrow F^\times \quad \text{defined by} \quad gk_xu \longmapsto x.$$

The set GB_2N is invariant under the left action of G and the right action of U. It is easy to see that every fiber $p^{-1}(x)$ is $G \times U$-invariant. By Bernstein's Localization Principle, it is enough to show that a distribution D on $p^{-1}(x)$ with the properties (4.2.65) and (4.2.66) vanishes.

We apply Proposition 4.3.2 of [3] to

$$G \times U_1 \cong G k_x U_1 = p^{-1}(x).$$

It shows that there exists a constant $c_2 \in \mathbb{C}$ such that

$$D(\phi) = c_2 \int\limits_G \int\limits_{U_1} \beta(g)\, \psi_{c_1,c_2}^{-1}(u_1)\, \phi(g \begin{bmatrix} 1 & & & \\ & x & & \\ & & x & \\ & & & 1 \end{bmatrix} u_1)\, du_1\, dg \tag{4.2.69}$$

for any $\phi \in \mathcal{S}(p^{-1}(x))$. Let $t \in F^\times$ be such that $\psi(c_2 2tx) \neq 1$,

$$n := \begin{bmatrix} 1 & & 2t & \\ & 1 & & -2ac^{-1}t \\ & & 1 & \\ & & & 1 \end{bmatrix} \in N_0 \subset G \text{ and } u := \begin{bmatrix} 1 & & 2tx & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}. \tag{4.2.70}$$

Hence, $\psi_{c_1,c_2}(u) = \psi(c_2 2tx) \neq 1$. Similarly as above, we calculate

$$\begin{aligned} D(L_{n^{-1}}\phi) &= c_2 \int\limits_G \int\limits_{U_1} \beta(g)\, \psi_{c_1,c_2}^{-1}(u_1)\, \phi(g n k_x u_1)\, du_1\, dg \\ &= c_2 \int\limits_G \int\limits_{U_1} \beta(g)\, \psi_{c_1,c_2}^{-1}(u_1)\, \phi(g k_x \begin{bmatrix} 1 & & 2tx & \\ & 1 & & -2ac^{-1}tx^{-1} \\ & & 1 & \\ & & & 1 \end{bmatrix} u_1)\, du_1\, dg. \end{aligned}$$

Hence,

$$\begin{aligned} &D(L_{n^{-1}}\phi) = \\ &= c_2 \int\limits_G \int\limits_F \int\limits_F \beta(g)\, \psi^{-1}(c_1 y)\, \phi(g k_x \begin{bmatrix} 1 & & 2tx & \\ & 1 & & -2ac^{-1}tx^{-1} \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ y & 1 & & z \\ & & 1 & -y \\ & & & 1 \end{bmatrix})\, dy\, dz\, dg \\ &= c_2 \int\limits_G \int\limits_F \int\limits_F \beta(g)\, \psi^{-1}(c_1 y)\, \phi(g \begin{bmatrix} 1 & & & -2txy \\ & 1 & -2txy & \\ & & 1 & \\ & & & 1 \end{bmatrix} k_x \begin{bmatrix} 1 & & & \\ y & 1 & & z \\ & & 1 & -y \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & 2tx & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}) dy\, dz\, dg \\ &= c_2 \int\limits_G \int\limits_F \int\limits_F \beta(g \begin{bmatrix} 1 & & & 2txy \\ & 1 & 2txy & \\ & & 1 & \\ & & & 1 \end{bmatrix})\, \psi^{-1}(c_1 y)\, \phi(g k_x \begin{bmatrix} 1 & & & \\ y & 1 & & z \\ & & 1 & -y \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & 2tx & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix})\, dy\, dz\, dg \end{aligned}$$

$$= c_2 \int\limits_{G} \int\limits_{U_1} \beta(g)\, \psi^{-1}(c_1 y)\, \phi(g k_x u_1 \begin{bmatrix} 1 & & 2tx & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix})\, du_1\, dg$$

$$= D(R_u \phi).$$

Hence, by (4.2.65) and (4.2.66), $D(\phi) = D(L_{n^{-1}}\phi) = D(R_u\phi) = \psi(c_2 2tx) D(\phi)$. It follows that $D(\phi) = 0$.

2.2) Now assume we are in the split case. Let $i \in \{1, 2\}$. We will show that a distribution D on Gg_iB_2N with the properties (4.2.65) and (4.2.66) must be zero. Calculations in coordinates verify that

$$g_i^{-1} G g_i \cap B_2 = \{ \begin{bmatrix} 1 & & & \\ \frac{y-1}{2r_i} & y & & \\ & & y & \frac{1-y}{2r_i} \\ & & & 1 \end{bmatrix} : y \in F^\times \}. \tag{4.2.71}$$

It follows that

$$Gg_iB_2N = Gg_iH_1N \sqcup Gg_i\tilde{g}_iN, \qquad \text{where } \tilde{g}_i = \begin{bmatrix} 1 & & & \\ -\frac{1}{2r_i} & 1 & & \\ & & 1 & \frac{1}{2r_i} \\ & & & 1 \end{bmatrix}, \tag{4.2.72}$$

and H_1 is as in (4.2.67). We will proceed to show that a distribution D on Gg_iB_2N with the properties (4.2.66) and

$$D(R_u\phi) = \psi(c_2 x) D(\phi) \qquad \text{for all } u = \begin{bmatrix} 1 & & x & y \\ & 1 & y & z \\ & & 1 & \\ & & & 1 \end{bmatrix} \in N, \tag{4.2.73}$$

must be zero.

2.2.1) We will first show that a distribution D on Gg_iH_1N with the properties (4.2.66) and (4.2.73) vanishes. We have

$$g_i^{-1} G g_i \cap H_1 N = \{ \begin{bmatrix} 1 & & -2r_i u & u \\ & 1 & u & v \\ & & 1 & \\ & & & 1 \end{bmatrix} : u, v \in F \}. \tag{4.2.74}$$

Hence

$$Gg_iH_1N = Gg_iH_1U_2, \qquad \text{where } U_2 = \begin{bmatrix} 1 & & * & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}. \tag{4.2.75}$$

In fact, every element of Gg_iH_1N can be written in the form gg_ik_xu with uniquely determined $x \in F^\times$ and $u \in U_2$. We consider the continuous map

$$p: \; Gg_iH_1N \longrightarrow F^\times \quad \text{ defined by } \quad gg_ik_xu \longmapsto x.$$

It is easy to see that every fiber $p^{-1}(x)$ is $G \times N$-invariant. By Bernstein's Localization Principle, it is enough to show that a distribution D on $p^{-1}(x)$ with the properties (4.2.66) and (4.2.73) vanishes. We apply Proposition 4.3.2 of [3] to

$$G \times U_2 \cong Gg_ik_xU_2 = p^{-1}(x).$$

It shows that there exists a constant $c_3 \in \mathbb{C}$ such that

$$D(\phi) = c_3 \int\limits_G \int\limits_F \beta(g)\, \psi^{-1}(c_2 z)\, \phi(gg_ik_x \begin{bmatrix} 1 & & z & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix})\, dz\, dg \tag{4.2.76}$$

for all $\phi \in \mathcal{S}(p^{-1}(x))$. Now, for any $y \in F$,

$$\begin{aligned} D(\phi) &= \\ &= c_3 \int\limits_G \int\limits_F \beta(g)\, \psi^{-1}(c_2 z)\, \phi(gg_ik_x \begin{bmatrix} 1 & & z & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & y \\ & 1 & y & \\ & & 1 & \\ & & & 1 \end{bmatrix}) dz dg \\ &= c_3 \int\limits_G \int\limits_F \beta(g)\, \psi^{-1}(c_2 z)\, \phi(gg_i \begin{bmatrix} 1 & & & y \\ & 1 & y & \\ & & 1 & \\ & & & 1 \end{bmatrix} k_x \begin{bmatrix} 1 & & z & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix})\, dz\, dg \\ &= c_3 \int\limits_G \int\limits_F \beta(g)\, \psi^{-1}(c_2 z)\, \phi(gg_i \begin{bmatrix} 1 & & -2r_iy & y \\ & 1 & y & \\ & & 1 & \\ & & & 1 \end{bmatrix} g_i^{-1} g_i \begin{bmatrix} 1 & & 2r_iy & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} k_x \begin{bmatrix} 1 & & z & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix})\, dz\, dg \end{aligned}$$

$$= c_3 \int\limits_G \int\limits_F \beta(g)\, \psi^{-1}(c_2 z)\, \phi(gg_i \begin{bmatrix} 1 & & 2r_i y & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} k_x \begin{bmatrix} 1 & & z & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix})\, dz\, dg$$

$$= c_3 \int\limits_G \int\limits_F \beta(g)\, \psi^{-1}(c_2 z)\, \phi(gg_i k_x \begin{bmatrix} 1 & & z+2r_i xy & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix})\, dz\, dg$$

$$= \psi(c_2 2r_i xy) c_3 \int\limits_G \int\limits_F \beta(g)\, \psi^{-1}(c_2 z)\, \phi(gg_i k_x \begin{bmatrix} 1 & & z & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix})\, dz\, dg$$

$$= \psi(c_2 2r_i xy) D(\phi).$$

It follows that $D(\phi) = 0$.

2.2.2) Finally, we will show that a distribution D on $Gg_i\tilde{g}_i N$ with the properties (4.2.66) and (4.2.73) vanishes. We have

$$(g_i\tilde{g}_i)^{-1} G g_i \tilde{g}_i \cap N = \{ \begin{bmatrix} 1 & & u & \\ & 1 & & v \\ & & 1 & \\ & & & 1 \end{bmatrix} : u, v \in F \}. \tag{4.2.77}$$

Hence

$$Gg_i\tilde{g}_i N = Gg_i\tilde{g}_i U_3, \qquad \text{where } U_3 = \begin{bmatrix} 1 & & & * \\ & 1 & * & \\ & & 1 & \\ & & & 1 \end{bmatrix}. \tag{4.2.78}$$

We apply Proposition 4.3.2 of [3] to

$$G \times U_3 \cong Gg_i\tilde{g}_i U_3.$$

It shows that there exists a constant $c_4 \in \mathbb{C}$ such that

$$D(\phi) = c_4 \int\limits_G \int\limits_F \beta(g)\, \phi(gg_i\tilde{g}_i \begin{bmatrix} 1 & & & z \\ & 1 & z & \\ & & 1 & \\ & & & 1 \end{bmatrix})\, dz\, dg \tag{4.2.79}$$

for any $\phi \in \mathcal{S}(Gg_i\tilde{g}_iN)$. Then, for any $x \in F$,

$$\begin{aligned}
\psi(c_2x)D(\phi) &= c_4 \int\limits_G \int\limits_F \beta(g)\, \phi(gg_i\tilde{g}_i \begin{bmatrix} 1&&&z\\&1&z&\\&&1&\\&&&1 \end{bmatrix} \begin{bmatrix} 1&&x&\\&1&&\\&&1&\\&&&1 \end{bmatrix})\, dz\, dg \\
&= c_4 \int\limits_G \int\limits_F \beta(g)\, \phi(gg_i\tilde{g}_i \begin{bmatrix} 1&&x&\\&1&&\\&&1&\\&&&1 \end{bmatrix} (g_i\tilde{g}_i)^{-1} g_i\tilde{g}_i \begin{bmatrix} 1&&&z\\&1&z&\\&&1&\\&&&1 \end{bmatrix})\, dz\, dg \\
&= c_4 \int\limits_G \int\limits_F \beta(g)\, \phi(gg_i\tilde{g}_i \begin{bmatrix} 1&&&z\\&1&z&\\&&1&\\&&&1 \end{bmatrix})\, dz\, dg \\
&= D(\phi).
\end{aligned}$$

It follows that $D(\phi) = 0$. This concludes the proof. $\square$

4.2.5 Some cases with no exceptional poles

The following is Theorem 4.2 of [9], with a slightly modified proof to accommodate our more general notion of exceptional pole.

Theorem 4.2.12. *Let (π, V) be an irreducible, admissible representation of* $\mathrm{GSp}(4, F)$ *admitting a (Λ, β)-Bessel model. Let μ be a character of $F^\times$. Assume that s_0 is an exceptional pole for the datum π, Λ, β, μ, as defined in the previous section. Then there exists a non-zero functional $\ell : V \to \mathbb{C}$ with the property*

$$\ell(\pi(g)v) = \mu^{-1}(\det(g))|\det(g)|^{-s_0-1/2}\, \ell(v) \qquad \text{for all } v \in V \text{ and } g \in G. \tag{4.2.80}$$

Proof. By definition, the function

$$\frac{Z(s, B, \Phi, \mu)}{L^{\mathrm{PS}}_\Lambda(s, \pi, \mu)} = \frac{Z(s, B, \Phi, \mu)}{L^{\mathrm{PS}}_{\mathrm{reg}}(s, \pi, \mu) L(s + 1/2, \Lambda_\mu)} \tag{4.2.81}$$

lies in $\mathbb{C}[q^s, q^{-s}]$, for any choice of $B \in \mathcal{B}(\pi, \Lambda, \beta)$ and $\Phi \in \mathcal{S}(V)$. In particular, we may evaluate at s_0. We note that

$$\left.\frac{Z(s, B, \Phi, \mu)}{L_\Lambda^{\mathrm{PS}}(s, \pi, \mu)}\right|_{s=s_0} = 0 \qquad \text{if } \Phi \in \mathcal{S}_0(V). \tag{4.2.82}$$

This follows from Lemma 4.2.6 i), and the fact that s_0 is a pole of $L(s+1/2, \Lambda_\mu)$. We now define

$$\ell(B) = \left.\frac{Z(s, B, \Phi_1, \mu)}{L_\Lambda^{\mathrm{PS}}(s, \pi, \mu)}\right|_{s=s_0}, \tag{4.2.83}$$

where, as before, Φ_1 is the characteristic function of $\mathfrak{o}_L \oplus \mathfrak{o}_L$. Since $Z(s, B, \Phi, \mu) = L_\Lambda^{\mathrm{PS}}(s, \pi, \mu)$ for some choice of B and Φ, equation (4.2.82) implies that ℓ is a non-zero functional. It follows from (4.2.28) that

$$Z(s, \pi(g)B, g.\Phi, \mu) = Z(s, B, \Phi, \mu)\mu^{-1}(\det(g))|\det(g)|^{-s-1/2} \qquad \text{for all } g \in G, \tag{4.2.84}$$

where $(g.\Phi)(x, y) = \Phi((x, y)g)$. Consequently,

$$\left.\frac{Z(s, \pi(g)B, g.\Phi_1, \mu)}{L_\Lambda^{\mathrm{PS}}(s, \pi, \mu)}\right|_{s=s_0} = \left.\frac{Z(s, B, \Phi_1, \mu)}{L_\Lambda^{\mathrm{PS}}(s, \pi, \mu)}\right|_{s=s_0} \mu^{-1}(\det(g))|\det(g)|^{-s_0-1/2}. \tag{4.2.85}$$

Since $g.\Phi - \Phi \in \mathcal{S}_0(V)$, property (4.2.82) allows us to replace $g.\Phi$ on the left hand side by Φ. It follows that ℓ has the asserted property (4.2.80). $\square$

Let $c_1, c_2 \in F^\times$. Recall from (4.2.64) the definition of the character ψ_{c_1,c_2} of U. An irreducible, admissible representation (π, V) of $\mathrm{GSp}(4, F)$ is called *generic* if it admits a non-zero functional $L : V \to \mathbb{C}$ satisfying

$$L(\pi(u)v) = \psi_{c_1,c_2}(u)L(v) \qquad \text{for all } v \in V,\ u \in U. \tag{4.2.86}$$

Such an L is called a ψ_{c_1,c_2}-Whittaker functional.

The proof of ii) of the following result has been sketched in Theorem 4.3 of [9]; here, we provide the details.

Corollary 4.2.13. *There are no exceptional poles for π, Λ, β, μ if one of the following conditions is satisfied.*

i) The character $\Lambda_\mu = \Lambda \cdot (\mu \circ N_{L/F})$ is ramified.

ii) π is generic.

Hence, in these cases we have $L_\Lambda^{\mathrm{PS}}(s, \pi, \mu) = L_{\mathrm{reg}}^{\mathrm{PS}}(s, \pi, \mu)$, and in particular the Piatetski-Shapiro L-factor is independent of the choice of Bessel model for π.

Proof. i) is immediate from Lemma 4.2.7 i).

ii) Let (π, V) be an irreducible, admissible, generic representation of $\mathrm{GSp}(4, F)$. Let $(\pi^\vee, V^\vee)$ be the contragredient representation. Then $\pi^\vee$ is also generic. Let L be a ψ_{c_1,c_2}-Whittaker functional on $V^\vee$.

Assume that π admits an exceptional pole; we will obtain a contradiction. By Theorem 4.2.12, there exists a character β of G and a functional $\ell : V \to \mathbb{C}$ such tht

$$\ell(\pi(g)v) = \beta(g)v \tag{4.2.87}$$

for all $v \in V$ and $g \in G$. We define a linear map

$$\Delta : \mathcal{S}(\mathrm{GSp}(4, F)) \longrightarrow V^\vee \tag{4.2.88}$$

by

$$\Delta(\phi)(v) = \int\limits_{\mathrm{GSp}(4,F)} \phi(g)\ell(\pi(g)v)\, dg, \tag{4.2.89}$$

where $\phi \in \mathcal{S}(\mathrm{GSp}(4, F))$, $v \in V$, and ℓ is a functional as in (4.2.80). Since ℓ is non-zero, it is easy to see that Δ is non-zero. One readily verifies that

$$\Delta(R_h\phi) = \pi^\vee(h)\Delta(\phi) \qquad \text{for all } h \in \mathrm{GSp}(4, F). \tag{4.2.90}$$

In particular, the image of Δ is an invariant subspace of $V^\vee$. Consequently, Δ is surjective. This allows us to define a non-zero distribution $D : \mathcal{S}(\mathrm{GSp}(4, F)) \to \mathbb{C}$ by

$$D(\phi) = L(\Delta(\phi)), \qquad \phi \in \mathcal{S}(\mathrm{GSp}(4, F)). \tag{4.2.91}$$

Since L is a ψ_{c_1,c_2}-Whittaker functional on $V^\vee$, it follows from (4.2.90) that

$$D(R_u\phi) = \psi_{c_1,c_2}(u)D(\phi) \qquad \text{for all } u \in U. \tag{4.2.92}$$

For $h \in G$, we have

$$\begin{aligned}
\Delta(L_h\phi)(v) &= \int\limits_{\mathrm{GSp}(4,F)} \phi(h^{-1}g)\ell(\pi(g)v)\,dg \\
&= \int\limits_{\mathrm{GSp}(4,F)} \phi(g)\ell(\pi(hg)v)\,dg \\
&= \beta(h) \int\limits_{\mathrm{GSp}(4,F)} \phi(g)\ell(\pi(g)v)\,dg
\end{aligned}$$

by (4.2.87). Hence $\Delta(L_h\phi) = \beta(h)\Delta(\phi)$, and thus

$$D(L_h\phi) = \beta(h)D(\phi) \qquad \text{for all } h \in G. \tag{4.2.93}$$

By Lemma 4.2.11, properties (4.2.92) and (4.2.93) imply that $D = 0$, which is a contradiction. $\square$

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